Algebraic State Space Representation of Logical Systems

Series One, Lesson Two

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Outline

1. Preliminaries

☞ Notations

- $\mathcal{D} := \{0, 1\}$, where $1 \sim T$ and $0 \sim F$.
- $\mathcal{D}_k := \{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\}$, $\mathcal{D}_2 = \mathcal{D}$.
- δ_n^i: the i-th column of the identity matrix I_n.
- $\Delta_k := \{\delta_k^1, \ldots, \delta_k^k\}$. Denote $\Delta := \Delta_2$.
- For a matrix $L \in \mathcal{M}_{m\times n}$, $Col_i(L) := L\delta_n^i$, $Row_j(L) := (\delta_m^j)^T L$, $Col(L) := \{Col_i(L), i = 1, \ldots, n\}$, $Row(L) := \{Row_j(L), j = 1, \ldots, m\}$.
- $\mathcal{L}_{n\times r} := \{L : L \in \mathcal{M}_{n\times r} \ and \ Col(L) \subset \Delta_n\}$. And any matrix $L \in \mathcal{L}_{n\times r}$ is called a logical matrix.
- If $L \in \mathcal{L}_{n\times r}$, then it is expressed and briefly denoted as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \ldots, \delta_n^{i_r}]$ and $L = \delta_n[i_1, i_2, \ldots, i_r]$.
- $\otimes$: the Kronecker product.

☞ Review of several matrix products

Traditional matrix product

Assume $A = (a_{ij}) \in M_{m\times n}, B = (b_{ij}) \in M_{n\times q}$, define the traditional product of matrices A and B as

$$AB = (c_{ij}) \in M_{m\times q} \tag{1}$$

where $c_{ij} = \sum_{i=1}^{n} a_{ik}b_{kj}, \quad i = 1, 2, \cdots m; j = 1, 2, \cdots q$.

Kronecker product

Assume $A = (a_{ij}) \in M_{m\times n}, B = (b_{ij}) \in M_{p\times q}$, then the Kronecker product of matrices A and B as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix} \in M_{mp\times nq}. \quad (2)$$

Property 1 of Kronecker product

If $A \in M_{m\times n}, B \in M_{p\times q}, C \in M_{n\times r}, D \in M_{q\times s}$, then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD),$$

particularly,

$$A \otimes B = (A \otimes I_p)(I_n \otimes B).$$

Property 2 of Kronecker product

1. Given two vectors $X \in \mathbb{R}^n, Y \in \mathbb{R}^n$, we have

$$V_c(XY^T) = Y \otimes X.$$

2. If $A \in M_{m\times p}, B \in M_{p\times q}, C \in M_{q\times n}$, then

$$V_c(ABC) = (C^T \otimes A)V_c(B).$$

$V_c(A) = (Col_1(A)^T, Col_2(A)^T, \dots, Col_p(A)^T)^T$.

Khatri-Rao product

Assume $n, m, p, q, n_i, m_j, p_i, q_j, (i = 1 \cdots r, j = 1 \cdots s)$ are all positive integer， and satisfy

$$\sum_{i=1}^{r} m_i = m, \sum_{j=1}^{s} n_j = n, \sum_{i=1}^{r} p_i = p, \sum_{j=1}^{s} q_j = q.$$

$A = (A_{ij}) \in M_{m\times n}, B = (B_{ij}) \in M_{p\times q}$ are block matrices, where $A_{ij} \in M_{m_i\times n_j}$, $B_{ij} \in M_{p_i\times q_j}$, that is,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1s} \\ B_{21} & B_{22} & \cdots & B_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ B_{r1} & B_{r2} & \cdots & B_{rs} \end{pmatrix}.$$

Define the Khatri-Rao product of matrices A and B as：

$$A * B = (A_{ij} \otimes B_{ij}) \in M_{u\times v}, \qquad (3)$$

where $u = \sum_{i=1}^{r} m_i p_i, v = \sum_{j=1}^{s} n_j q_j$.

Remark about Khatri-Rao product

1. If $r = s = 1$, then $A * B = A \otimes B$;
2. If $A \in M_{m \times r}$, $B \in M_{n \times r}$, $A = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \end{pmatrix}$, $B = \begin{pmatrix} B_1 & B_2 & \cdots & B_r \end{pmatrix}$, then

$$A * B = \begin{pmatrix} A_1 \otimes B_1 & A_2 \otimes B_2 & \cdots & A_r \otimes B_r \end{pmatrix}.$$

Hadamard product

Assume $A = (a_{ij}), B = (b_{ij}) \in M_{m\times n}$, then the Hadamard product of A and B is defined as

$$A \odot B = (a_{ij}b_{ij}) \in M_{m\times n}. \tag{4}$$

Particularly, if

$$m = p, n = q, m_1 = m_2 = \cdots = m_r = n_1 = n_2 = \cdots = n_s = 1$$

in the definition of Khatri-Rao product, then $A * B = A \odot B$.

☞ Semi-tensor product

Definition 1

Let $A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{p\times q}, t = lcm(n,p)$. Then the semi-tensor product (STP) of A and B is

$$A \ltimes B := (A \otimes I_{t/n})(B \otimes I_{t/p}),$$

where $lcm(n,p)$ represents the least common multiple of n and p.

Problem 1

In Definition 1, what does happen to the product above if t is any other common multiple of n and p?

☞ Pseudo-Commutativity

Lemma 1

(1) Given a matrix $A \in \mathcal{M}$ and a column vector $x \in \mathcal{V}_t$. Then

$$x \ltimes A = (I_t \otimes A)x,$$

$$A \ltimes x = x(I_t \otimes A).$$

(2) Given two column vectors $x \in \mathcal{V}_n, y \in \mathcal{V}_m$. Then

$$W_{[n,m]} \ltimes x \ltimes y = y \ltimes x,$$

$$x \ltimes y \ltimes W_{[n,m]} = y \ltimes x,$$

where $W_{[n,m]}$ is a swap matrix.

2. Propositional logic

☞ Propositions

Example 1

Consider the following statements.

1. A dog has 4 legs;
2. The snow is black;
3. There is another human in the universe.
4. Bridge, stream, village.

It is not hard to find that statement 1 is “true” and statement 2 is “false”. Statement 3 may be “true” or “false”, although we still do not know the answer. Thus, statements 1-3 are all propositions. Statement 4 is not a proposition, because neither “true” nor “false” is meaningfully applied to it.

☞ Logical operators

- Negation $\neg$ (否). The negation of proposition A, denoted by $\neg A$, is its opposite. A is true if and only if $\neg A$ is false, and vice versa.
- Conjunction $\wedge$(合取). The conjunction of A and B, denoted by $A \wedge B$, is a proposition that is true only if A and B are true.
- Disjunction $\vee$(析取). The disjunction of A and B, denoted by $A \vee B$, is a proposition that is true if at least one of A and B is true.
- Conditional $\rightarrow$(蕴涵). The conditional of A and B, denoted by $A \rightarrow B$, means that A implies B, i.e., if A then B.
- Biconditional $\leftrightarrow$(等值). The biconditional of A and B, denoted by $A \leftrightarrow B$, means that A is true if and only if B is true.
- $\cdots\cdots$

☞ Logical function

Definition 2

1. A logical variable is a variable which takes value from $\mathcal{D}$.

2. A set of logical variables $x_1, \ldots, x_n$ are independent, if for any fixed values $x_j, j \neq i$, the logical variable x_i can still take value either 1 or 0.

3. A logical function of logical variable $x_1, \ldots, x_n$ is a logical expression involving $x_1, \ldots, x_n$ and some possible statements (called constants), joined by logical operators.

Hence, a logical function is mapping $f : \mathcal{D}^n \to \mathcal{D}$. It is also called an n-ary operator.

Example 2

$f(p, q, r) = (\neg p) \to (q \vee r)$ is a logical function of p, q, r.

Table 1: Truth table of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \bar{\nabla}, \uparrow$ and $\downarrow$

x	y	$\neg x$	$\neg y$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\bar{\nabla}$	$\uparrow$	$\downarrow$
1	1	0	0	1	1	1	1	0	0	0
1	0	0	1	0	1	0	0	1	1	0
0	1	1	0	0	1	1	0	1	1	0
0	0	1	1	0	0	1	1	0	1	0

- $\bar{\nabla}$ is logical operator “exclusive or”(EOR 异或);
- $\uparrow$ is “not and”(NAND 与非);
- $\downarrow$ is “not or”(NOR 或非).

Problem 2

Is there any other $2-$ary logical operator? How many?

☞ Normal Form

Definition 3

Let $\{p_1, p_2, \ldots, p_n\}$ be a set of logical variables. Define a set of logical variables by also including their negations, as follows:

$$P := \{p_1, \neg p_1, p_2, \neg p_2, \ldots, p_n, \neg p_n\}.$$

1. If $c := \bigwedge_{i=1}^{s} a_i,\ a_i \in P$, then c is called a basic conjunctive form.

2. If $d := \bigvee_{i=1}^{s} a_i,\ a_i \in P$, then d is called a basic disjunctive form.

3. If $m := \bigvee_{i=1}^{s} c_i$, where c_i are basic conjunctive form, then m is called a disjunctive form.

4. If $n := \bigwedge_{i=1}^{s} d_i$, where d_i are basic disjunctive form, then m is called a conjunctive form.

Lemma 2

Any logical expression can be expressed in disjunctive normal form as well as conjunctive normal form. (about the proof, see page 12 of [1])

Example 3

Consider $f(p,q,r) = ((p \vee q) \to \neg r) \to ((r \to p) \wedge (r \vee q))$.

Its disjunctive normal form is:

$$f = (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r),$$

and its conjunctive normal form is:

$$f = (\neg p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee q \vee r).$$

[1] D. Z. Cheng, H. S. Qi, Z. Q. Li, Analysis and Control of Boolean Networks: A Semi-tensor Product Approach, London, Springer, 2011.

3. Structure matrix of a logical function

☞ Structure matrix of a logical operator

To use matrix expression of logic, we identify

$$1 \sim \delta_2^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \;\; 0 \sim \delta_2^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and call them the vector forms of logic values. Then in vector form, an n-ary logical function f becomes a mapping $f : \Delta^n \to \Delta$. That is,

$$\mathcal{D} \sim \Delta,$$

Similarly,

$$f : \mathcal{D}^n \to \mathcal{D} \Leftrightarrow f : \Delta^n \to \Delta.$$

Definition 5

Let $f(x_1, \ldots, x_n)$ be an n-ary logical function. $L_f \in \mathcal{L}_{2\times 2^n}$ is called the structure matrix of f, if in vector form we have

$$f(x_1, \ldots, x_n) = L_f \ltimes_{i=1}^{n} x_i. \tag{5}$$

The structure matrices of some fundamental operators:

$$
\begin{aligned}
M_{\neg} &:= M_n = \delta_2[2\ 1], \\
M_{\vee} &:= M_d = \delta_2[1\ 1\ 1\ 2], \\
M_{\wedge} &:= M_c = \delta_2[1\ 2\ 2\ 2], \\
M_{\rightarrow} &:= M_i = \delta_2[1\ 2\ 1\ 1], \\
M_{\leftrightarrow} &:= M_e = \delta_2[1\ 2\ 2\ 1].
\end{aligned}
$$

To check the first one

Check the vector equation $\neg p = M_n p$, where p is the vector form of a logical variable, and

$$M_n = \delta_2[2\ 1] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

1) When $p = T$, we have $p = T \sim \delta_2^1 \Longrightarrow M_n p = \delta_2^2 \sim F$.

2) When $p = F$, we have $p = F \sim \delta_2^2 \Longrightarrow M_n p = \delta_2^1 \sim T$.

☞ Power-reducing matrices

Lemma 3

Given a logical variable $x \in \Delta$. Then

$$x^2 = M_r x,$$

where $M_r := \delta_4[1\ 4]$ is called the power-reducing matrix.

To check Lemma 3

1) When $x = \delta_2^1$, then

$$x^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = M_r x,$$

2) When $x = \delta_2^2$, the proof is omitted.

Problem 3

If $x \in \Delta_k$, write the corresponding power-reducing matrix $M_{r,k}$.

Definition 6

A matrix $L \in \mathcal{L}_{n\times r}$ is called logical matrix, if $Col_i(L) \in \Delta_n$ for any $1 \leq i \leq r$.

Lemma 4

1. A swap matrix is a logical matrix, i.e., $W_{[m,n]} \in \mathcal{L}_{mn\times mn}$;
2. The identity matrix is a logical matrix, i.e., $I_m \in \mathcal{L}_{m\times m}$;
3. The power-reducing matrix is a logical matrix, i.e., $M_r \in \mathcal{L}_{4\times 2}$;
4. If $L \in \mathcal{L}$, then $L \otimes I_n \in \mathcal{L}, \quad I_n \otimes L \in \mathcal{L}$;
5. If $A, B \in \mathcal{L}$, then $A \ltimes B \in \mathcal{L}$. (For any $A \in R_{m\times n}$, we have $A\delta_n^i = Col_i(A)$)

Proposition 1

Let $f(x_1,\ldots,x_n)$ be an n-ary logical function with logical variables $x_1,\ldots,x_n$. Then, f can be expressed as

$$f(x_1,\ldots,x_n) = \ltimes_i \xi_i,$$

where $\xi_i \in \{M_n, M_d, M_c, x_1, \ldots, x_n\}$.

Proof outline of Proposition 1

1. disjunctive (conjunctive) form;

2. the structure matrices of logical operators $\vee$, $\wedge$ and $\neg$.

Example 4

Let $f(p,q,r) = (p \wedge \neg q) \vee (r \wedge p)$. Then in vector form we have

$$\begin{aligned} f(p,q,r) &= (p \wedge \neg q) \vee (r \wedge p) \\ &= M_d(M_c p(M_n q))(M_c rp) \\ &= M_d M_c p M_n q M_c rp. \end{aligned}$$

Based on Proposition 1, we have

Proposition 2

Let $f(x_1, \ldots, x_n)$ be an n-ary logical function. Then there exists a unique structure matrix $L_f \in \mathcal{L}_{2\times 2^n}$ such that (5) holds.

Proof outline of Proposition 2

1. Using $x_i M = (I \otimes M)x_i$ to move all variables x_i to the rear;
2. Using swap matrix to change the order of two variables x_i and x_j;
3. Using power-reducing matrix to reduce the powers of x_i to 1;
4. Prove $L_f \in \mathcal{L}_{2\times 2^n}$;
5. Prove the uniqueness.

Example 5

In Example 4 we have already have $f(p,q,r) = M_d M_c p M_n q M_c r p$. We continue by converting this into canonical form:

$$
\begin{aligned}
f(p,q,r) &= M_d M_c p M_n q M_c r p \\
&= M_d M_c (I_2 \otimes M_n) p q M_c r p \\
&= M_d M_c (I_2 \otimes M_n)(I_4 \otimes M_c) p q r p \\
&= M_d M_c (I_2 \otimes M_n)(I_4 \otimes M_c) p W_{[2,4]} p q r \\
&= M_d M_c (I_2 \otimes M_n)(I_4 \otimes M_c)(I_2 \otimes W_{[2,4]}) p^2 q r \\
&= M_d M_c (I_2 \otimes M_n)(I_4 \otimes M_c)(I_2 \otimes W_{[2,4]}) M_r p q r \\
&= M_f p q r.
\end{aligned}
$$

Remark 1

Disjunctive (conjunctive) form is not necessary.

Example 6

Let $f(x,y) = (x \vee y) \to (x \wedge y)$. Then in vector form we have

$$\begin{aligned} f(x,y) &= (x \vee y) \to (x \wedge y) \\ &= M_i M_d xy M_c xy \\ &= M_i M_d (I_4 \otimes M_c) xyxy \\ &= M_i M_d (I_4 \otimes M_c) x W_{[2]} x y^2 \\ &= M_i M_d (I_4 \otimes M_c)(I_2 \otimes W_{[2]}) x^2 y^2. \end{aligned}$$

Using the power-reducing matrix, we have

$$\begin{aligned} f(x,y) &= M_i M_d (I_4 \otimes M_c)(I_2 \otimes W_{[2]}) x^2 y^2 \\ &= M_i M_d (I_4 \otimes M_c)(I_2 \otimes W_{[2]}) M_r x M_r y \\ &= M_i M_d (I_4 \otimes M_c)(I_2 \otimes W_{[2]}) M_r (I_2 \otimes M_r) xy. \end{aligned}$$

Thus, $L = M_i M_d (I_4 \otimes M_c)(I_2 \otimes W_{[2]}) M_r (I_2 \otimes M_r)$ is the structure matrix of $f(x,y)$.

☞ Another computation of structure matrix

Example 7

Let $f(p,q,r) = (\neg p) \to (q \vee r)$. The truth table is derived about logical function $f(p,q,r)$.

Table 2: Truth table of $f(p,q,r)$

p	q	r	$\neg p$	$q \vee r$	f
1	1	1	0	1	1
1	1	0	0	1	1
1	0	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	1	0	1	1	1
0	0	1	1	1	1
0	0	0	1	0	0

Definition 4

Let $f(x_1, \ldots, x_n)$ be an n-ary logical function. Denote the column of f in its truth table by T_f, and call it the truth vector of f.

Computation of structure matrix L_f

$$Row_1(L_f) := T_f^T,$$

$$Row_2(L_f) := \neg Row_1(L_f),$$

where $\neg Row_1(L_f)$ is derived by taking negation on each elements of $Row_1(L_f)$, and T_f is the truth vector of logical operator f.

Example 7 (continuing)

From Table 2, we have the truth vector of f

$$T_f = [1\ 1\ 1\ 1\ 1\ 1\ 1\ 0]^T.$$

Thus the structure matrix of f is obtained

$$\begin{aligned} L_f &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \delta_2\, [1, 1, 1, 1, 1, 1, 1, 2] \end{aligned}$$

Notice that the value order of all variables in the truth table can not be changed when using truth vector to deduce the structure matrix.

Example 7 (continuing)

In vector form, we have

$$
\begin{aligned}
f(p,q,r) &= L_f(p \ltimes q \ltimes r) \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} p \ltimes q \ltimes r
\end{aligned}
$$

Table 3: Truth table of $f(p,q,r)$

(p,q,r)	$p \ltimes q \ltimes r$	f
$(1,1,1) \sim (\delta_2^1, \delta_2^1, \delta_2^1)$	δ_8^1	$1 \sim \delta_2^1$
$(1,1,0) \sim (\delta_2^1, \delta_2^1, \delta_2^2)$	δ_8^2	$1 \sim \delta_2^1$
$(1,0,1) \sim (\delta_2^1, \delta_2^2, \delta_2^1)$	δ_8^3	$1 \sim \delta_2^1$
$(1,0,0) \sim (\delta_2^1, \delta_2^2, \delta_2^2)$	δ_8^4	$1 \sim \delta_2^1$
$(0,1,1) \sim (\delta_2^2, \delta_2^1, \delta_2^1)$	δ_8^5	$1 \sim \delta_2^1$
$(0,1,0) \sim (\delta_2^2, \delta_2^1, \delta_2^2)$	δ_8^6	$1 \sim \delta_2^1$
$(0,0,1) \sim (\delta_2^2, \delta_2^2, \delta_2^1)$	δ_8^7	$1 \sim \delta_2^1$
$(0,0,0) \sim (\delta_2^2, \delta_2^2, \delta_2^2)$	δ_8^8	$0 \sim \delta_2^2$

☞ Dummy matrices

The dummy matrices are defined as

$$M_u = \delta_2[1\ 1\ 2\ 2], \ \ M_v = \delta_2[1\ 2\ 1\ 2].$$

Lemma 5

In vector form we have

$$M_u xy = x, \ \ M_v xy = y.$$

Check Lemma 5

1) When $x = \delta_2^1$, then

$$M_u xy = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} y = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} y = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x,$$

2) When $x = \delta_2^2$, then

$$M_u xy = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} y = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x,$$

Therefore, $M_u xy = x$ is proved.

3) No matter $x = \delta_2^1$ or $x = \delta_2^2$, we all have $M_v x = I_2$. Hence, it is easy to have $M_v xy = y$.

Problem 4

In Lemma 5, if vectors x and y are all in Δ_k, then write the dummy matrices.

Example 8

Consider logical functions

$$\begin{cases} f_1(x_1, x_2, x_3) = x_1 \wedge x_2, \\ f_2(x_1, x_2, x_3) = x_2 \vee x_3. \end{cases}$$

Using dummy matrices, the vector forms of $f_1(x_1, x_2, x_3)$ and $f_2(x_2, x_2, x_3)$ can be expressed as

$$\begin{cases} f_1(x_1, x_2, x_3) = M_c x_1 x_2 = M_c x_1 M_u x_2 x_3 = M_c(I_2 \otimes M_u) x_1 x_2 x_3, \\ f_2(x_1, x_2, x_3) = M_d x_2 x_3 = M_d M_v x_1 x_2 x_3. \end{cases}$$

☞ Algebraic form⇒ Logical form

How can we get the logical form for a given algebraic form?

$$f(x_1, x_2, \cdots, x_n) = M_f \ltimes_{i=1}^n x_i.$$

Algorithm 1

Let $f(x_1, x_2, \cdots, x_n)$ be logical form. Its algebraic form is

$$f(x_1, x_2, \cdots, x_n) = M_f \ltimes_{i=1}^n x_i,$$

where

$$M_f = \delta_2[a_1, a_2, \cdots, a_{2^n}].$$

Its logical form can be calculated as follows:

- Step 1: Split the structure matrix into 2^{n-1} equal blocks as

$$\begin{aligned} M_f &= [\delta_2[a_1, a_2], \delta_2[a_3, a_4], \cdots, \delta_2[a_{2^n-1}, a_{2^n}]] \\ &:= [L_1, L_2, \cdots, L_{2^{n-1}}]. \end{aligned}$$

- Step 2: For every $j = \{1,\ 2,\ \dots,\ 2^{n-1}\}$, factorize $\delta_{2^{n-1}}^j$ as

$$\delta_{2^{n-1}}^j = \delta_2^{\alpha_1^j} \delta_2^{\alpha_2^j} \cdots \delta_2^{\alpha_{n-1}^j}.$$

- Step 3: The logical (disjunctive normal) form of $f(x_1, x_2, \cdots, x_n)$ is constructed as

$$f(x_1, x_2, \cdots, x_n) = \bigvee_{j=1}^{2^{n-1}} \Big[\bigwedge_{i=1}^{n-1} \lambda_i^j x_i \bigwedge \phi_j(x_n) \Big],$$

where

$$\lambda_i^j x_i = \begin{cases} x_i, & \alpha_i^j = 1 \\ \neg x_i, & \alpha_i^j = 2, \end{cases} \qquad \phi_j(x_n) = \begin{cases} 1, & L_j = \delta_2[1, 1] \\ x_n, & L_j = \delta_2[1, 2] \\ \neg x_n, & L_j = \delta_2[2, 1] \\ 0, & L_j = \delta_2[2, 2]. \end{cases}$$

Remark 2

Logical expression of a logical function is not unique.

Example 9

Given

$$f(x_1, x_2, x_3) = (x_1 \bar{\vee} x_2) \to (\neg x_2 \leftrightarrow x_3),$$

find its disjunctive normal form. Derive its structure matrix first:

$$M_f = \delta_2[1, 1, 1, 2, 2, 1, 1, 1].$$

Then

$$\begin{aligned} f(x_1, x_2, x_3) = \quad & [x_1 \wedge x_2 \wedge \phi_1(x_3)] \vee \\ & [x_1 \wedge \neg x_2 \wedge \phi_2(x_3)] \vee \\ & [\neg x_1 \wedge x_2 \wedge \phi_3(x_3)] \vee \\ & [\neg x_1 \wedge \neg x_2 \wedge \phi_4(x_3)]. \end{aligned}$$

Example 9 (continuing)

According to Algorithm 1, we have

$$\phi_1 = \phi_4 = 1, \quad \phi_2 = x_3, \quad \phi_3 = \neg x_3.$$

Thus we have

$$\begin{aligned} f(x_1, x_2, x_3) = \quad & [x_1 \wedge x_2] \vee \\ & [x_1 \wedge \neg x_2 \wedge x_3] \vee \\ & [\neg x_1 \wedge x_2 \wedge \neg x_3] \vee \\ & [\neg x_1 \wedge \neg x_2]. \end{aligned}$$

4. Algebraic expression of logical systems

Definition 7

A static logical equations is expressed as

$$\begin{cases} f_1(x_1,\ldots,x_n)=c_1, \\ f_2(x_1,\ldots,x_n)=c_2, \\ \qquad \vdots \\ f_m(x_1,\ldots,x_n)=c_m, \end{cases} \tag{6}$$

where f_i is a logical function, x_i is a logical argument (unknown), and c_i is a logical constant.

A set of logical constants $d_i, i=1,\ldots,n$, which makes $x_i=d_i$ satisfying (6), is said to be a solution of logical equations (6).

Lemma 6

Assume

$$\begin{cases} y = M_y \ltimes_{i=1}^{n} x_i, \\ z = M_z \ltimes_{i=1}^{n} x_i, \end{cases}$$

where $x_i \in \Delta, i = 1, 2, \ldots, n$, $M_y \in \mathcal{L}_{2\times 2^n}$ and $M_z \in \mathcal{L}_{2\times 2^n}$. Then

$$yz = (M_y * M_z) \ltimes_{i=1}^{n} x_i,$$

where $M_y * M_z := [Col_1(M_y) \otimes Col_1(M_z), \ldots, Col_{2^n}(M_y) \otimes Col_{2^n}(M_z)]$.

Remark 3

Result in Lemma 6 can be extend to multiple case or logical equations (6).

Proof outline of Lemma 6

1. $yz = M_y \ltimes_{i=1}^{n} x_i M_z \ltimes_{i=1}^{n} x_i = M_y(I_{2^n} \otimes M_z)M_{r,2^n} \ltimes_{i=1}^{n} x_i = M_{yz} \ltimes_{i=1}^{n} x_i$, where $M_{yz} \in \mathcal{L}_{4\times 2^n}$.

2. Assuming $\ltimes_{i=1}^{n} x_i = \delta_{2^n}^{r}$ ($1 \leq r \leq 2^n$ is arbitrary), have $y = Col_r(M_y)$ and $z = Col_r(M_z)$.

3. $Col_r(M_{yz}) = yz = Col_r(M_y) \ltimes Col_r(M_z) = Col_r(M_y) \otimes Col_r(M_z)$.

Example 10

Consider the following logical equations

$$\begin{cases} x_1 \wedge x_2 = 0, \\ x_2 \vee x_3 = 1, \\ x_3 \leftrightarrow x_1 = 1. \end{cases} \tag{7}$$

Denote $x = x_1 x_2 x_3$. Then the vector form of each equation is expressed as

$$\begin{cases} L_1 x = M_c (I_2 \otimes M_u) x = \delta_2^2, \\ L_2 x = M_d M_v x = \delta_2^1, \\ L_3 x = M_e W_{[2]} M_u x = \delta_2^1. \end{cases}$$

According to Lemma 6, the vector form of equations (7) is

$$Lx = (L_1 * L_2 * L_3) x = \delta_8^5.$$

Logical dynamic systems

- Boolean network (BN)

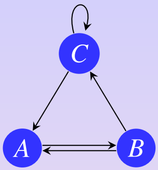

Figure 1: BN

$$\begin{cases} A(t+1) = B(t) \wedge C(t) \\ B(t+1) = \neg A(t) \\ C(t+1) = B(t) \vee C(t) \end{cases}$$

- Boolean control network (BCN)

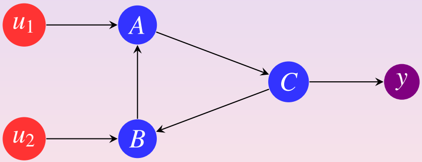

Figure 2: BCN

$$\begin{cases} A(t+1) = B(t) \wedge u_1(t) \\ B(t+1) = C(t) \vee u_2(t) \\ C(t+1) = A(t) \end{cases}$$

$$y(t) = \neg C(t)$$

- Boolean network

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), x_i \in \mathcal{D}, \end{cases} \tag{8}$$

- Boolean control network

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)), \\ y_j(t) = h_j(x(t)), j = 1, \cdots, p, \end{cases} \tag{9}$$

where $x_i, u_i, y_i \in \mathcal{D}$.

Proposition 3 (Algebraic form of dynamic logical network)

- Boolean network (8) has algebraic form

$$x(t+1) = Lx(t), \tag{10}$$

  where $x(t) = \ltimes_{i=1}^{n} x_i(t)$, $L \in \mathcal{L}_{2^n \times 2^n}$.

- Boolean control network (9) has algebraic form

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ y(t) = Hx(t), \end{cases} \tag{11}$$

  where $x(t) = \ltimes_{i=1}^{n} x_i(t)$, $u(t) = \ltimes_{i=1}^{m} u_i(t)$, $y(t) = \ltimes_{i=1}^{p} y_i(t)$, $L \in \mathcal{L}_{2^n \times 2^{n+m}}$, $H \in \mathcal{L}_{2^p \times 2^n}$.

Proof outline of BN case in Proposition 3

1. $x_i(t+1) = L_i \ltimes_{i=1}^n x_i$.

2. $\ltimes_{i=1}^n x_i(t+1) = (L_1 \ltimes_{i=1}^n x_i) \ltimes (L_2 \ltimes_{i=1}^n x_i) \ltimes \cdots \ltimes (L_n \ltimes_{i=1}^n x_i) := L \ltimes_{i=1}^n x_i$.

3. Assuming $\ltimes_{i=1}^n x_i = \delta_{2^n}^r (1 \leq r \leq n)$, have $x_i(t+1) = Col_r(L_i)$.

4. $Col_r(L) = \ltimes_{i=1}^n x_i(t+1) = \ltimes_{i=1}^n Col_r(L_i) = \otimes_{i=1}^n Col_r(L_i)$.

Example 11

- Consider the Boolean network in Figure 1, we have

$$L = \delta_8 \begin{bmatrix} 3 & 7 & 7 & 8 & 1 & 5 & 5 & 6 \end{bmatrix}.$$

- Consider the Boolean control network in Figure 2, we have

$$\begin{aligned} L &= \delta_8[1\ 1\ 5\ 5\ 2\ 2\ 6\ 6\ 1\ 3\ 5\ 7\ 2\ 4\ 6\ 8 \\ &\quad 5\ 5\ 5\ 5\ 6\ 6\ 6\ 6\ 5\ 7\ 5\ 7\ 6\ 8\ 6\ 8]; \\ H &= \delta_2[2\ 1\ 2\ 1\ 2\ 1\ 2\ 1]. \end{aligned}$$

☞ General logical networks I

- k-valued logical networks

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), \end{cases} \quad (12)$$

  where $x_i \in \mathcal{D}_k$.

- k-valued logical network (12) has algebraic form

$$x(t+1) = Lx(t), \quad (13)$$

  where $x(t) = \ltimes_{i=1}^{n} x_i(t)$, $L \in \mathcal{L}_{k^n \times k^n}$.

☞ General logical networks II

- Mix-valued logical networks

$$\begin{cases} x_1(t+1) = f_1(x_1(t), \cdots, x_n(t)) \\ \vdots \\ x_n(t+1) = f_n(x_1(t), \cdots, x_n(t)), \end{cases} \tag{14}$$

  where $x_i \in \mathcal{D}_{k_i}$.

- Mix-valued logical network (14) has algebraic form

$$x(t+1) = Lx(t), \tag{15}$$

  where $x(t) = \ltimes_{i=1}^{n} x_i(t)$, $L \in \mathcal{L}_{k^n \times k^n}$ and $k = \prod_{i=1}^{n} k_i$.

☞ Appendix 1

Defined the k-value power-reducing matrix as

$$M_{r,k} = \begin{bmatrix} \delta_k^1 & 0 & \cdots & 0 \\ 0 & \delta_k^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \delta_k^k \end{bmatrix} = diag\{\delta_k^1, \delta_k^2, \cdots, \delta_k^k\}$$

Lemma 7

Given a logical variable $x \in \Delta_{r,k}$. Then

$$x^2 = M_{r,k}x.$$

Proof of Lemma 7

Assume $x = \delta_k^i$, then it is easy to have

$$x^2 = \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} i-1 \\ 1 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} k-i \end{bmatrix} \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} i-1 \\ 1 \\ \left.\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right\} k-i \end{bmatrix} = \begin{bmatrix} \left.\begin{matrix} 0_k \\ \vdots \\ 0_k \end{matrix}\right\} i-1 \\ \delta_k^i \\ \left.\begin{matrix} 0_k \\ \vdots \\ 0_k \end{matrix}\right\} k-i \end{bmatrix},$$

Proof of Lemma 7 (continuning)

$$M_{r,k}x = \begin{bmatrix} \delta_k^1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \delta_k^2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \delta_k^i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \delta_k^k \end{bmatrix} \delta_k^i = \begin{bmatrix} \left.\begin{matrix} 0_k \\ \vdots \\ 0_k \end{matrix}\right\} i-1 \\ \delta_k^i \\ \left.\begin{matrix} 0_k \\ \vdots \\ 0_k \end{matrix}\right\} k-i \end{bmatrix}.$$

☞ Appendix 2

A swap matrix of dimension (m,n)-is defined as follows:

$$
\begin{aligned}
W_{[m,n]} &:= \left[I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \cdots, I_n \otimes \delta_m^m \right] \\
&= \left[\delta_n^1 \ltimes \delta_m^1 \cdots \delta_n^n \ltimes \delta_m^1 \cdots \delta_n^1 \ltimes \delta_m^m \cdots \delta_n^n \ltimes \delta_m^m \right] \\
&= \begin{bmatrix} I_m \otimes (\delta_n^1)^T \\ \vdots \\ I_m \otimes (\delta_n^n)^T \end{bmatrix}.
\end{aligned}
$$

Swap matrices have many properties, see pp:38-41 of reference [1].

For example

$$
W_{[2,3]} = \begin{array}{c} \begin{array}{cccccc} (11) & (12) & (13) & (21) & (22) & (23) \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right) \end{array} \begin{array}{c} \\ (11) \\ (21) \\ (12) \\ (22) \\ (13) \\ (23) \end{array}
$$

5. Example and Homework

Example 12

A says “B is a liar”, B says “C is a liar”, C says “Both A and B are liars”. Who is a liar?

Denote $T \sim 1 \sim \delta_2^1$ and $D \sim 0 \sim \delta_2^2$ representing honest and not being honest, respectively.

1. Define three logical variables: p: A is honest, or not; q: B is honest, or not; r: c is honest, or not.

2. Have $p \leftrightarrow \neg q$; $q \leftrightarrow \neg r$; $r \leftrightarrow (\neg p \wedge \neg q)$.

3. The problem is equivalent to when the following logical equation has solution.

$$(p \leftrightarrow \neg q) \wedge (q \leftrightarrow \neg r) \wedge (r \leftrightarrow (\neg p \wedge \neg q)) = 1.$$

Example (continuing)

4 The algebraic form of the logical equation above is

$$M_c^2 M_e p M_n q M_e q M_n r M_e r M_n p M_n q = \delta_2^1.$$

5 Via computing, derive that $Lpqr = \delta_2^1$ with

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

6 It is clear that the algebraic equation above has unique solution:

$$p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; r = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

that is to say, A and C are both liars, only B is honest.

Homework

1. Answer Problems 1-4 above.

2. Prove the algebraic forms of k (or mix) valued logical networks.

3. Given an algebraic form Lx with $x = \ltimes_{i=1}^{n} x_i$ and $L \in \mathcal{L}_{2^n \times 2^n}$.

   a. Can we derive its logical conjunctive normal form?

   b. How to get it?

6. References

[1] D. Z. Cheng, H. S. Qi, Z. Q. Li, Analysis and Control of Boolean Networks: A Semi-tensor Product Approach, London, Springer, 2011.

[2] D. Z. Cheng, H. S. Qi, Y. Zhao, An introduction to semi-tensor product of matrices and its application, Singapore, World Scientific, 2012.

[3] 程代展，齐洪胜，矩阵的半张量积: 理论与应用，北京，科学出版社，2007.

[4] http://lsc.amss.ac.cn/ dcheng/ (Publications, STP Toolbox, Courses, Preprint)

Thanks for your attention!

Q & A