Set Controllability, Observability and Output Tracking of Boolean Networks

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Outline

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Introduction

Boolean Networks

- ➥ With the rapid development of systems biology and medical science, Boolean networks (BNs) become an active research topic in biology, physics and engineering.
- ➥ The control of BNs is important for the disease treatment and pharmaceutical preparation [*].
- ➥ A major goal of systems biology is to develop suitable mathematical tools for the analysis and control of complex biological systems [*].

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[*] T. Ideker, et al., A new approach to decoding life: Systems biology, Annu. Rev. Genomics Hum. Genet., 2001, 2: 343-372.

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Logical Networks

- Classification of logical networks: BNs, multi-valued logical networks, mix-valued logical networks.
- Applications of logical networks: circuit design, finite automata, game theory, graph theory, fuzzy control, feedback register, and so on.
- Existing methods: computer simulation, polynomial theory over finite field, semi-tensor product of matrices.

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Logical Networks

Consider the following Boolean model of signal transduction networks*:

$$
\begin{cases}
x_1(t+1) = x_8(t), \\
x_2(t+1) = x_1(t), \\
x_3(t+1) = x_2(t), \\
x_4(t+1) = x_8(t), \\
x_5(t+1) = x_4(t), \\
x_6(t+1) = x_3(t) \vee x_5(t), \\
x_7(t+1) = x_8(t), \\
x_8(t+1) = x_6(t) \wedge \neg x_7(t),
\end{cases} \tag{1}
$$

where x_1 stands for the nitric oxide synthase (NOS), x_2 represents the nitric oxide (NO), x_3 is the guanyl cyclase (GC), x_4 is the phospholipase C (PLC), x_5 represents the inositol-1,4,5-trisphosphate (InsP3), x_6 is the Ca^{2+} influx to the cytosol from intracellular stores (CIS), x_7 stands for the Ca^{2+}ATPases and Ca^{2+}/H^+ antiporters responsible for Ca^{2+} efflux from the cytosol (Ca^{2+}ATPase), and x_8 is the cytosolic Ca^{2+} increase (Ca_c^{2+}).

*A. Saadatpour, I. Albert, R. Albert, Attractor analysis of asynchronous Boolean models of signal transduction networks, J. Theoret. Biol., 2010, 266: 641-656.

Introduction

Logical Networks

The "minimal" Boolean model for the lactose operon in Escherichia coli* is given as follows:

$$\begin{cases} x_1(t+1) = \neg u_1(t) \wedge (x_3(t) \vee u_2(t)), \\ x_2(t+1) = x_1(t), \\ x_3(t+1) = \neg u_1(t) \wedge [(x_2(t) \wedge u_2(t)) \\ \qquad\qquad \vee (x_3(t) \wedge \neg x_2(t))], \end{cases} \tag{2}$$

where $x_1 \in \mathcal{D}$ denotes the mRNA, $x_2 \in \mathcal{D}$ the lacZ polypeptide, $x_3 \in \mathcal{D}$ the intracellular lactose, $u_1 \in \mathcal{D}$ the external glucose, and $u_2 \in \mathcal{D}$ the external lactose.

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*R. Robeva, T. Hodge, Mathematical Concepts and Methods in Modern Biology: Using Modern Discrete Models, Academic Press, 2013.

Introduction

Logical Networks

Consider a networked evolutionary game (NEG) consisting of four players, in which the set of players is denoted by $N = \{P_1, P_2, P_3, P_4\}$ and the network graph of the game is string. The neighborhood of each P_i is denoted by $U(i)$. The basic game of this NEG is the Rock-Scissors-Paper game, whose payoff matrix is given in Table 1, where "Rock", "Scissors" and "Paper" are denoted by "1", "2" and "3", respectively. Hence, all the players have the same set of strategies: $S = \{1, 2, 3\}$.

Table 1: Payoff Matrix.

$P_1 \setminus P_2$	1	2	3
1	(0, 0)	(1, -1)	(-1, 1)
2	(-1, 1)	(0, 0)	(1, -1)
3	(1, -1)	(-1, 1)	(0, 0)

Introduction

Logical Networks

Suppose that the game can repeat infinitely. At each time, P_i only plays the Rock-Scissors-Paper game with its neighbors in $U(i)$, and its aggregate payoff $c_i : S^{|U(i)|} \to \mathbb{R}$ is the sum of payoffs gained by playing with all its neighbors in $U(i)$, that is,

$$c_i(P_i, P_j | j \in U(i)) = \sum_{j \in U(i)} c_{ij}(P_i, P_j), \tag{3}$$

where $c_{ij} : S \times S \to \mathbb{R}$ denotes the payoff of P_i playing with its neighbor $P_j, j \in U(i)$.
The strategy updating rule is: For each $i = 1, 2$, $P_i(t+1)$ is updated by the best strategy from strategies of its neighbors in $U(i)$ at time t. Precisely, if $j^* = \arg\max_{j \in U(i)} c_j(P_j, P_k | k \in U(j))$, then $P_i(t+1) = P_{j^*}(t)$. When the neighbors with maximum payoff are not unique, say, $\arg\max_{j \in U(i)} c_j(P_j, P_k | k \in U(j)) := \{j_1^*, \cdots, j_r^*\}$, we choose $j^* = \min\{j_1^*, \cdots, j_r^*\}$.
According to the strategy updating rule, we obtain the following 3-valued logical network:

$$P_i(t+1) = f_i(P_1(t), P_2(t), P_3(t), P_4(t)), \tag{4}$$

where $f_i, i = 1, 2, 3, 4$ are 3-valued logical functions, which can be uniquely determined by the strategy updating rule.

Introduction

Semi-Tensor Product of Matrices

- Prof. Daizhan Cheng developed the semi-tensor product (STP) of matrices for the analysis and control of logical networks[*].
- The STP of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}), \tag{5}$$

  where $\alpha = lcm(n, p)$ denotes the least multiple of n and p.
- The main advantage of STP: converting a logical network into a (bi)linear form, which makes a bridge between logical networks and classic control theory.

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[*] D. Cheng, H. Qi, Z. Li, Analysis and Control of Boolean Networks: A Semi-tensor Product Approach, London: Springer, 2011.

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Semi-Tensor Product of Matrices

- Given a logical function $f : \mathcal{D}^s \mapsto \mathcal{D}$. There exists a unique structural matrix $M_f \in \mathcal{L}_{2\times 2^s}$ such that

$$f(x_1, x_2, \cdots, x_s) = M_f \ltimes_{i=1}^{s} x_i, \quad x_i \in \Delta := \{\delta_2^1, \delta_2^2\}. \quad (6)$$

- An algebraic state space representation approach is established for logical networks. * *

* E. Fornasini, M. Valcher, Recent developments in Boolean networks control, Journal of Control and Decision, 2016, 3(1): 1-18.

* J. Lu, H. Li, Y. Liu, F. Li, Survey on semi-tensor product method with its applications in logical networks and other finite-valued systems, IET Control Theory & Applications, 2017, 11(13): 2040-2047.

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Semi-Tensor Product of Matrices

Algebraic form of (1) is

$$x(t+1) = Lx(t),$$

where

$$
\begin{array}{rl}
L = \delta_{256}[& 2\ 148\ \ 1\ 147\ \ 2\ 148\ \ 2\ 148\ \ 2\ 148\ \ 1\ 147\ \ 2\ 148\ \ 2\ 148 \\
& 10\ 156\ \ 9\ 155\ \ 10\ 156\ \ 10\ 156\ \ 10\ 156\ \ 9\ 155\ \ 10\ 156\ \ 10\ 156 \\
& 2\ 148\ \ 1\ 147\ \ 2\ 148\ \ 2\ 148\ \ 6\ 152\ \ 5\ 151\ \ 6\ 152\ \ 6\ 152 \\
& 10\ 156\ \ 9\ 155\ \ 10\ 156\ \ 10\ 156\ \ 14\ 160\ \ 13\ 159\ \ 14\ 160\ \ 14\ 160 \\
& 34\ 180\ \ 33\ 179\ \ 34\ 180\ \ 34\ 180\ \ 34\ 180\ \ 33\ 179\ \ 34\ 180\ \ 34\ 180 \\
& 42\ 188\ \ 41\ 187\ \ 42\ 188\ \ 42\ 188\ \ 42\ 188\ \ 41\ 187\ \ 42\ 188\ \ 42\ 188 \\
& 34\ 180\ \ 33\ 179\ \ 34\ 180\ \ 34\ 180\ \ 38\ 184\ \ 37\ 183\ \ 38\ 184\ \ 38\ 184 \\
& 42\ 188\ \ 41\ 187\ \ 42\ 188\ \ 42\ 188\ \ 46\ 192\ \ 45\ 191\ \ 46\ 192\ \ 46\ 192 \\
& 66\ 212\ \ 65\ 211\ \ 66\ 212\ \ 66\ 212\ \ 66\ 212\ \ 65\ 211\ \ 66\ 212\ \ 66\ 212 \\
& 74\ 220\ \ 73\ 219\ \ 74\ 220\ \ 74\ 220\ \ 74\ 220\ \ 73\ 219\ \ 74\ 220\ \ 74\ 220 \\
& 66\ 212\ \ 65\ 211\ \ 66\ 212\ \ 66\ 212\ \ 70\ 216\ \ 69\ 215\ \ 70\ 216\ \ 70\ 216 \\
& 74\ 220\ \ 73\ 219\ \ 74\ 220\ \ 74\ 220\ \ 78\ 224\ \ 77\ 223\ \ 78\ 224\ \ 78\ 224 \\
& 98\ 244\ \ 97\ 243\ \ 98\ 244\ \ 98\ 244\ \ 98\ 244\ \ 97\ 243\ \ 98\ 244\ \ 98\ 244 \\
& 106\ 252\ \ 105\ 251\ \ 106\ 252\ \ 106\ 252\ \ 106\ 252\ \ 105\ 251\ \ 106\ 252\ \ 106\ 252 \\
& 98\ 244\ \ 97\ 243\ \ 98\ 244\ \ 98\ 244\ \ 102\ 248\ \ 101\ 247\ \ 102\ 248\ \ 102\ 248 \\
& 106\ 252\ \ 105\ 251\ \ 106\ 252\ \ 106\ 252\ \ 110\ 256\ \ 109\ 255\ \ 110\ 256\ \ 110\ 256].
\end{array}
$$

Introduction

Semi-Tensor Product of Matrices

➥ The main advantage of STP: By the STP method, a logical expression can be converted into a linear (bilinear) form.

➥ Recent Results:

✔ Analysis and Control of BNs: Controllability, Observability, Realization, Identification, Disturbance Decoupling, Optimal Control, etc.;

✔ Model Generalization*: Delayed Logical Networks, Probabilistic Logical Networks, Asynchronous Logical Networks, Switched Logical Networks, Logical Networks with Impulsive Effect.

✔ Applications*: Fault Detection of Circuits, Graph Coloring, Game Theory, Fuzzy Control, Finite Automata, Nonlinear Feedback Shift Register, Smart Grid, Vehicle Control, etc.

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*H. Li, G. Zhao, P. Guo, Z. Liu, Analysis and Control of Finite-Value Systems, CRC Press, 2018.

*H. Li, G. Zhao, M. Meng, J. Feng, A survey on applications of semi-tensor product method in engineering, Science China Information Sciences, 2018, 61(1): 010202.

Introduction

Controllability and Observability of BNs

- Controllability and observability of BNs are two fundamental properties*.
- Controllability means the reachability from any initial state to any terminal state, which provides a reachable set approach to the control design of BNs*.
- Observability means to distinguish any two different initial states from a piece of output trajectories, which is important for identification and observer design.

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*D. Cheng, H. Qi, Controllability and observability of Boolean control networks, Automatica, 2009, 45(7): 1659-1667.

*R. Li, M. Yang, T. Chu, State feedback stabilization for probabilistic Boolean networks, Automatica, 2014, 50(4): 1272-1278.

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*R. Li, M. Yang, T. Chu, State feedback stabilization for probabilistic Boolean networks, Automatica, 2014, 50(4): 1272-1278.

Introduction

Recent Works on Controllability

- ✔ Y. Zhao, H. Qi, D. Cheng, Input-state incidence matrix of Boolean control networks and its applications, *Systems & Control Letters*, 2010, 59(12): 767-774.
  ✔ D. Laschov, M. Margaliot, Controllability of Boolean control networks via the Perron-Frobenius theory, *Automatica*, 2012, 48(6): 1218-1223.
  ✔ Q. Zhu, Y. Liu, J. Lu, J. Cao, Further results on the controllability of Boolean control networks, *IEEE Transactions on Automatic Control*, 2019, 64(1): 440-442.
  ✔ H. Li, Y. Wang, Controllability analysis and control design for switched Boolean networks with state and input constraints, *SIAM Journal on Control and Optimization*, 2015, 53(5): 2955-2979.
  ✔ Y. Liu, H. Chen, B. Wu, Controllability of Boolean control networks with impulsive effects and forbidden states, *Mathematical Methods in the Applied Sciences*, 2014, 37(1): 1-9.
  ✔ Y. Liu, H. Chen, J. Lu, B. Wu, Controllability of probabilistic Boolean control networks based on transition probability matrices, *Automatica*, 2015, 52: 340-345.
  ✔ J. Lu, J. Zhong, D. W. C. Ho, Y. Tang, J. Cao, On controllability of delayed Boolean control networks, *SIAM Journal on Control and Optimization*, 2016, 54(2): 475-494.

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Recent Works on Observability

➥ ✔ E. Fornasini, M. Valcher, Observability, reconstructibility and state observers of Boolean control networks, *IEEE Transactions on Automatic Control*, 2013, 58(6): 1390-1401.

✔ D. Laschov, M. Margaliot, G. Even, Observability of Boolean networks: A graph-theoretic approach, *Automatica*, 2013, 49: 2351-2362.

✔ R. Li, M. Yang, T. Chu, Observability conditions of Boolean control networks, *International Journal of Robust and Nonlinear Control*, 2014, 24: 2711-2723.

✔ D. Cheng, H. Qi, T. Liu, Y. Wang, A note on observability of Boolean control networks, *Systems & Control Letters*, 2016, 87: 76-82.

✔ Y. Guo, W. Gui, C. Yang, Redefined observability matrix for Boolean networks and distinguishable partitions of state space, *Automatica*, 2018, 91: 316-319.

✔ Y. Yu, M. Meng, J. Feng, Observability of Boolean networks via matrix equations, *Automatica*, 2020, 111: 108621.

✔ K. Zhang, L. Zhang, Observability of Boolean control networks: A unified approach based on finite automata, *IEEE Transactions on Automatic Control*, 2016, 61(9): 2733-2738.

✔ D. Cheng, C. Li, F. He, Observability of Boolean networks via set controllability approach, *Systems & Control Letters*, 2018, 115: 22-25.

✔ Y. Guo, Observability of Boolean control networks using parallel extension and set reachability, *IEEE Transactions on Neural Networks and Learning Systems*, 2018, 29(12): 6402-6408.

Introduction

Set Controllability of BNs

- Set controllability depicts the reachability from the family of initial sets to the family of destination sets*.
- Applications of set controllability: observability, set stabilization, partial stabilization, output tracking, synchronization, consensus, etc*.
- The key of applying set controllability is to properly construct the family of initial sets and the family of destination sets.

*D. Cheng, C. Li, F. He, Observability of Boolean networks via set controllability approach, Systems & Control Letters, 2018, 115: 22-25.

*J. Wang, Y. Liu, H. Li, Finite-time controllability and set controllability of impulsive probabilistic Boolean control networks, IEEE Access, 2020, 8: 111995-112002.

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Introduction

Recent Works on Output Tracking

- ✔ E. Fornasini, M. Valcher, Feedback stabilization, regulation and optimal control of Boolean control networks, *Proc. 2014 American Control Conf.*, 2014, pp. 1981-1986.
  ✔ H. Li, Y. Wang, L. Xie, Output tracking control of Boolean control networks via state feedback: Constant reference signal case, *Automatica*, 2015, 59: 54-59.
  ✔ H. Li, L. Xie, Y. Wang, Output regulation of Boolean control networks, *IEEE Transactions on Automatic Control*, 2017, 62(6): 2993-2998.
  ✔ J. Zhong, D. W. C. Ho, J. Lu, Q. Jiao, Pinning controllers for activation output tracking of Boolean network under one-bit perturbation, *IEEE Transactions on Cybernetics*, 2018, 49(9): 3398-3408.
  ✔ X. Zhang, Y. Wang, D. Cheng, Output tracking of Boolean control networks, *IEEE Transactions on Automatic Control*, 2020, 65(6): 2730-2735.
  ✔ Z. Zhang, T. Leifeld, P. Zhang, Finite horizon tracking control of Boolean control networks, *IEEE Transactions on Automatic Control*, 2018, 63(6): 1798-1805.

Introduction

Notations

1. $\mathcal{D} := \{1, 0\}$.
2. $\Delta_n := \{\delta_n^k : k = 1, \cdots, n\}$, where δ_n^k denotes the k-th column of I_n. $\Delta := \Delta_2$.
3. An $n \times t$ logical matrix $M = [\delta_n^{i_1}\ \delta_n^{i_2}\ \cdots\ \delta_n^{i_t}]$ is briefly denoted by $M = \delta_n[i_1\ i_2\ \cdots\ i_t]$. The set of $n \times t$ logical matrices is denoted by $\mathcal{L}_{n\times t}$.
4. Given $A \in \mathbb{R}^{m\times n}$, $Col_i(A)$, $Row_i(A)$ and $(A)_{i,j}$ denote the i-th column, the i-th row and the (i,j)-th element of A, respectively. $Col_i(A) = A\delta_n^i$, $Row_i(A) = (\delta_m^i)^\top A$.
5. Given $A \in \mathbb{R}^{n\times mp}$, denote the i-th $n \times p$ block of A by $Blk_i(A)$.
6. $\neg$, $\vee$ and $\wedge$ denote “Negation”, “Disjunction” and “Conjunction”, respectively.

Outline

1. Introduction
2. **Set Controllability of Boolean Networks**
3. Observability of Boolean Networks
4. Output Tracking of Boolean Networks
5. Concluding Remarks

Set Controllability of BNs

Boolean Control Networks

- ◇ Consider the following Boolean control network (BCN):

$$\begin{cases} X(t+1) = f(U(t), X(t)), \\ Y(t) = h(X(t)). \end{cases} \tag{7}$$

- ✔ $X(t) = (x_1(t), \cdots, x_n(t)) \in \mathcal{D}^n$, $U(t) = (u_1(t), \cdots, u_m(t)) \in \mathcal{D}^m$, and $Y(t) = (y_1(t), \cdots, y_p(t)) \in \mathcal{D}^p$ denote the states, control inputs, and outputs of system (7) at time t, respectively.
- ✔ $f : \mathcal{D}^{m+n} \to \mathcal{D}^n$ and $h : \mathcal{D}^n \to \mathcal{D}^p$ are logical mappings.

Set Controllability of BNs

Definition of Controllability

Definition 1 (D. Cheng et al., Springer, 2011)

Consider BCN (7) and given $X_0, X_d \in \mathcal{D}^n$.

1. X_d is said to be reachable from X_0 at the T-th step, $T \in \mathbb{Z}_+$, if one can find a control sequence $\{U(t) \in \mathcal{D}^m : t = 0, \cdots, T-1\}$ such that $X(T; X_0, U) = X_d$;

2. X_d is said to be reachable from X_0, if there exists $T \in \mathbb{Z}_+$ such that X_d is reachable from X_0 at the T-th step;

3. BCN (7) is said to be controllable at X_0, if for any given $X_d \in \mathcal{D}^n$, X_d is reachable from X_0;

4. BCN (7) is said to be controllable, if it is controllable at any $X_0 \in \mathcal{D}^n$.

Set Controllability of BNs

Algebraic Form of BCNs

Using the vector form of Boolean variables, that is, $2-i \sim \delta_2^i$, $i=1,2$, BCN (7) can be converted into the following algebraic form:

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ y(t) = Hx(t), \end{cases} \quad (8)$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t)$, $u(t) = \ltimes_{i=1}^{m} u_i(t)$, $y(t) = \ltimes_{i=1}^{p} y_i(t)$, $L \in \mathcal{L}_{2^n \times 2^{n+m}}$ is the state transition matrix, and $H \in \mathcal{L}_{2^p \times 2^n}$ is the output matrix.

Set Controllability of BNs

Controllability Matrix

- Controllability matrix:

$$\mathcal{C} := \sum_{t=1}^{2^n} M^t.$$

where $M := \sum_{i=1}^{2^m} L\delta_{2^m}^i$ is the one-step state transition matrix.

Remark 1

When defining the controllability matrix, the matrix addition and matrix product can be replaced by Boolean addition $(+_{\mathcal{B}})$ and Boolean product $(\times_{\mathcal{B}})$, respectively[a].

[a] D. Cheng, C. Li, F. He, Observability of Boolean networks via set controllability approach, *Syst. Control Lett.*, 2018, 115: 22-25.

Set Controllability of BNs

Controllability Criterion

Theorem 1 (Y. Zhao et al., Syst. Control Lett., 2010)

Consider BCN (8).

(i). $x_d = \delta_{2^n}^{\varphi}$ is reachable from $x_0 = \delta_{2^n}^{\theta}$ at the T-th step, $T \in \mathbb{Z}_+$, if and only if $(M^T)_{\varphi,\theta} > 0$.

(ii). $x_d = \delta_{2^n}^{\varphi}$ is reachable from $x_0 = \delta_{2^n}^{\theta}$, if and only if $(\mathcal{C})_{\varphi,\theta} > 0$.

(iii). System (8) is controllable at $x_0 = \delta_{2^n}^{\theta}$, if and only if $Col_{\theta}(\mathcal{C}) > 0$.

(iv). System (8) is controllable, if and only if $\mathcal{C} > 0$.

Set Controllability of BNs

Proof of (i). in Theorem 1

- ✔ When $T=1$, by Definition 1, $x_d=\delta_{2^n}^{\varphi}$ is reachable from $x_0=\delta_{2^n}^{\theta}$ at the first step, if and only if there exists $u(0):=\delta_{2^m}^{\mu}$ such that $\delta_{2^n}^{\varphi}=L\delta_{2^m}^{\mu}\delta_{2^n}^{\theta}$. Then, we have $(M)_{\varphi,\theta}=\sum_{i=1}^{2^m}(L\delta_{2^m}^{i})_{\varphi,\theta}\geq(L\delta_{2^m}^{\mu})_{\varphi,\theta}=1>0$.
- ✔ Assume that the conclusion holds for $T=s\geq 1$. When $T=s+1$, since a path from x_0 to x_d at the $(s+1)$-th step can always be considered as a path from x_0 to $\bar{x}_0=\delta_{2^n}^{\gamma}$ at the s-th step and then from $\bar{x}_0$ to x_d at the first step, we have $(M^s)_{\gamma,\theta}>0$ and $(M)_{\varphi,\gamma}>0$. Therefore, $(M^{s+1})_{\varphi,\theta}=\sum_{k=1}^{2^n}(M)_{\varphi,k}(M^s)_{k,\theta}\geq(M)_{\varphi,\gamma}(M^s)_{\gamma,\theta}>0$.
- ✔ By induction, the conclusion is true for any $T\in\mathbb{Z}_+$.

Set Controllability of BNs

Definition of Set Controllability

- Given the family of initial sets P_0 and the family of destination sets P_d respectively as follows:

$$P_0 := \{s_1^0, s_2^0, \cdots, s_\alpha^0\} \subseteq 2^{\Delta_{2^n}}, \; s_j^0 \neq \emptyset, j = 1, 2, \cdots, \alpha;$$
$$P_d := \{s_1^d, s_2^d, \cdots, s_\beta^d\} \subseteq 2^{\Delta_{2^n}}, \; s_i^d \neq \emptyset, i = 1, 2, \cdots, \beta.$$

Definition 2 (D. Cheng et al., Syst. Control Lett., 2018)

BCN (8) with P_0 and P_d is

1. set controllable from $s_j^0 \in P_0$ to $s_i^d \in P_d$, if there exist $x_0 \in s_j^0$ and $x_d \in s_i^d$, such that x_d is reachable from x_0;
2. set controllable at s_j^0, if for any $s_i^d \in P_d$, it is set controllable from s_j^0 to s_i^d;
3. set controllable, if it is set controllable at any $s_j^0 \in P_0$.

Set Controllability of BNs

Set Controllability Matrix

- Set controllability matrix:

$$\mathcal{C}_S := J_d^\top \mathcal{C} J_0.$$

✔ J_0 (J_d) is the initial (destination) index matrix defined as

$$J_0 := [V(s_1^0)\ V(s_2^0)\ \cdots\ V(s_\alpha^0)],$$
$$J_d := [V(s_1^d)\ V(s_2^d)\ \cdots\ V(s_\beta^d)].$$

✔ $V(s) \in \mathbb{R}_{2^n}$ with $(V(s))_i := \begin{cases} 1, & \delta_{2^n}^i \in s, \\ 0, & \delta_{2^n}^i \notin s \end{cases}$ is the index vector of $s \subseteq \Delta_{2^n}$.

Set Controllability of BNs

Set Controllability Criterion

Theorem 2 (D. Cheng et al., Syst. Control Lett., 2018)

BCN (8) with P_0 and P_d is

(i). set controllable from s_j^0 to s_i^d, if and only if $(\mathcal{C}_S)_{i,j} > 0$.

(ii). set controllable at s_j^0, if and only if $Col_j(\mathcal{C}_S) > 0$.

(iii). set controllable, if and only if $\mathcal{C}_S > 0$.

Remark 2

Controllability is a spacial kind of set controllability. If

$$P_0 = P_d = \left\{\{\delta_{2^n}^1\}, \cdots, \{\delta_{2^n}^{2^n}\}\right\},$$

the set controllability degrades to the controllability defined in Definition 1.

Set Controllability of BNs

An Example

Given $P_0 = \left\{\{\delta_8^2, \delta_8^3\}, \{\delta_8^1, \delta_8^5, \delta_8^7\}\right\}$ and $P_d = \left\{\{\delta_8^4\}, \{\delta_8^2, \delta_8^8\}\right\}$. Consider the set controllability of BCN (8) with $L = \delta_8[3\ 5\ 1\ 3\ 1\ 5\ 6\ 3\ 7\ 4\ 1\ 3\ 1\ 5\ 6\ 3]$

- ◇ Controllability matrix:

$$\mathcal{C} := \sum_{t=1}^{8}\left(\sum_{i=1}^{2} L\delta_2^i\right)^t = \begin{bmatrix} 230 & 170 & 110 & 220 & 110 & 220 & 120 & 220 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 55 & 92 & 110 & 62 & 110 & 60 & 120 & 62 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 60 & 73 & 120 & 48 & 120 & 50 & 100 & 48 \\ 110 & 84 & 60 & 120 & 60 & 120 & 50 & 120 \\ 55 & 90 & 110 & 60 & 110 & 60 & 120 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- ◇ Initial and destination index matrices:

$$J_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^\top, \quad J_d = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

- ◇ Set controllability matrix:

$$\mathcal{C}_S = J_d^\top \mathcal{C} J_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, BCN (8) is not set controllable with respect to P_0 and P_d.

Outline

Observability of BNs

Definition of Observability

Definition 3 (Y. Zhao et al., Syst. Control Lett., 2010)

BCN (8) is observable, if for any two distinct states $x_0, \bar{x}_0$, there exists a control sequence $\{u_0, u_1, \cdots, u_{p-1}\}, p \in \mathbb{Z}_+$, such that the corresponding output sequences $(y_0, y_1, \cdots, y_p) \neq (\bar{y}_0, \bar{y}_1, \cdots, \bar{y}_p)$.

Remark 3

K. Zhang and L. Zhang 2016[a] discussed four different definitions of observability and pointed out that Definition 3 is the most sharp one. Thus, we take Definition 3 as the standard one and concentrate on this definition. An interesting topic is to explore other definitions via set controllability approach.

[a] K. Zhang, L. Zhang, Observability of Boolean control networks: A unified approach based on finite automata, *IEEE Trans. Aut. Contr.*, 2016, 61(9): 2733-2738.

Observability of BNs

Construct Dual System

- ◇ Split the product state space $\Delta_{2^n} \times \Delta_{2^n}$ into a partition of three components as

$$D = \{zx : z = x\}, \ \Theta = \{zx : z \neq x, Hz = Hx\}, \ \Xi = \{zx : z \neq x, Hz \neq Hx\}.$$

- ◇ Using algebraic form (8), construct a dual system as

$$\begin{cases} z(t+1) = Lu(t)z(t), \\ x(t+1) = Lu(t)x(t). \end{cases} \tag{9}$$

Remark 4

By classifying the pairs of states, Cheng et al. 2016[a] proposed an effective criterion for observability via constructing observability matrix. However, in order to obtain the observability matrix, one needs to proceed an iterative algorithm.

[a] D. Cheng, et al., A note on observability of Boolean control networks, *Syst. Control Lett.*, 2016, 87: 76-82.

Observability of BNs

Observability Criterion

- Construct the family of initial sets P_0 and the family of destination sets P_d respectively as follows:

$$P_0 := \bigcup_{zx \in \Theta} \left\{ \{zx\} \right\}; \tag{10}$$

$$P_d := \{\Xi\}. \tag{11}$$

Theorem 3 (D. Cheng et al., Syst. Control Lett., 2018)

BCN (8) is observable, if and only if system (9) is set controllable with respect to P_0 and P_d, which are defined in (10) and (11), respectively.

Observability of BNs

An Example

Example 1

Consider the reduced model for the lac operon in the bacterium Escherichia coli[a]:

$$\begin{cases} x_1(t+1) = \neg u_1(t) \wedge (x_2(t) \vee x_3(t)), \\ x_2(t+1) = \neg u_1(t) \wedge u_2(t) \wedge x_1(t), \\ x_3(t+1) = \neg u_1(t) \wedge (u_2(t) \vee (u_3(t) \wedge x_1(t))), \end{cases} \tag{12}$$

where x_1, x_2 and x_3 represent lac mRNA, lactose in high and medium concentrations, respectively; u_1, u_2 and u_3 are extra-cellular glucose, high and medium extracellular lactose, respectively.

[a] A. Veliz-Cuba, B. Stigler, Boolean models can explain bistability in the lac operon, *J. Comput. Biol.*, 2011, 18(6): 783-794.

Observability of BNs

An Example

- ◇ Assume that the outputs are

$$\begin{cases} y_1(t) = x_1(t) \wedge \neg x_2(t) \wedge x_3(t), \\ y_2(t) = (x_1(t) \wedge \neg x_3(t)) \vee \{\neg x_1(t) \\ \qquad\qquad \vee [x_2(t) \vee (\neg x_2(t) \wedge \neg x_3(t))]\}, \\ y_3(t) = (x_1(t) \wedge \neg x_2(t) \wedge x_3(t)) \vee (\neg x_1(t) \wedge x_3(t)). \end{cases} \tag{13}$$

- ◇ The algebraic form of BCN (12) with outputs (13) is

$$\begin{cases} x(t+1) = Lu(t)x(t), \\ y(t) = Hx(t), \end{cases}$$

where

$$L = \delta_8[8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 1\ 1\ 1\ 5\ 3\ 3\ 3\ 7\ 1\ 1\ 1\ 5\ 3\ 3\ 3\ 7\ 3\ 3\ 3\ 7\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8]$$

and

$$H = \delta_8[8\ 6\ 3\ 6\ 5\ 6\ 7\ 6].$$

Observability of BNs

An Example

- It is easy to figure out that

$$\begin{aligned}\Theta &= \{\delta_{64}^{12}, \delta_{64}^{14}, \delta_{64}^{16}, \delta_{64}^{26}, \delta_{64}^{30}, \delta_{64}^{32}, \delta_{64}^{42}, \delta_{64}^{44}, \delta_{64}^{48}, \delta_{64}^{58}, \delta_{64}^{60}, \delta_{64}^{62}\},\\ \Xi &= \Delta_{64} \setminus \Big(\Theta \cup \{\delta_{64}^{1}, \delta_{64}^{10}, \delta_{64}^{19}, \delta_{64}^{28}, \delta_{64}^{37}, \delta_{64}^{46}, \delta_{64}^{55}, \delta_{64}^{64}\}\Big).\end{aligned}$$

- Construct a dual system as

$$\begin{cases} z(t+1) = Lu(t)z(t),\\ x(t+1) = Lu(t)x(t).\end{cases} \tag{14}$$

Setting $w(t) = z(t)x(t)$, we have

$$w(t+1) = \bar{L}u(t)w(t),$$

where $\bar{L} = \delta_{64}[64\ 64\ 64\ \cdots\ 60\ 60\ 60\ 64]$.

- The controllability matrix of system (14) can be calculated by

$$\bar{\mathcal{C}} := \sum_{j=1}^{64}\Big(\sum_{i=1}^{8} \bar{L}\delta_8^i\Big)^j.$$

Observability of BNs

An Example

- Construct the family of initial sets P_0 and the family of destination sets P_d respectively as follows:

$$P_0 := \bigcup_{zx \in \Theta} \Big\{ \{zx\} \Big\}, \ P_d := \{\Xi\}.$$

- According to P_0 and P_d, we have

$$J_0 = \delta_{64}[12\ 14\ 16\ 26\ 30\ 32\ 42\ 44\ 48\ 58\ 60\ 62], \ J_d = \sum_{\delta_{64}^i \in \Xi} \delta_{64}^i.$$

- Since

$$\mathcal{C}_S = J_d^\top \bar{\mathcal{C}} J_0 > 0,$$

system (14) is set controllable with respect to P_0 and P_d.

- By virtue of Theorem 3, BCN (12) with outputs (13) is observable.

Outline

Output Tracking of BNs

Definition of Output Tracking

Definition 4 (H. Li et al., Automatica, 2015)

Given a constant reference signal $y^* \in \Delta_{2^p}$. BCN (8) is trackable with respect to y^*, if for any initial state $x_0 \in \Delta_{2^n}$, there exist $T \in \mathbb{Z}_+$ and a control sequence $\{u(t) : t \in \mathbb{N}\}$, such that $y(t; x_0, u) = y^*$ holds for any integer $t \geq T$.

- ◇ For the given reference signal $y^* = \delta_{2^p}^{\alpha}$, define

$$\mathcal{O}(\alpha) := \{\delta_{2^n}^i \in \Delta_{2^n} : Col_i(H) = y^*\}, \tag{15}$$

and presuppose $\mathcal{O}(\alpha) \neq \emptyset$.

Output Tracking of BNs

Output Tracking Criterion-Reachable Set Approach

◇ For $S \subseteq \Delta_{2^n}$, $S \neq \emptyset$ and $k \in \mathbb{Z}_+$, define

$$\mathcal{R}_k(S) = \Big\{ x_0 \in \Delta_{2^n} : \text{there exists } \{u(t) \in \Delta_{2^m} : t = 0, \cdots, k-1\} \text{ such that } x(k; x_0, u) \in S \Big\}.$$

Theorem 4 (H. Li et al., Automatica, 2015)

The output of BCN (8) tracks the reference signal $y^* = \delta_{2^p}^{\alpha}$ by a state feedback control, if and only if there exist a nonempty set $S \subseteq \mathcal{O}(\alpha)$ and an integer $1 \leq \tau \leq 2^n$, such that

$$\begin{cases} S \subseteq \mathcal{R}_1(S), \\ \mathcal{R}_\tau(S) = \Delta_{2^n}. \end{cases} \tag{16}$$

Output Tracking of BNs

Output Tracking Criterion-Reachable Set Approach

- Consider BCN (8) with $L = \delta_{2^n}[i_1\ i_2\ \cdots\ i_{2^{m+n}}]$. Suppose that there exist a nonempty set $S \subseteq \mathcal{O}(\alpha)$ and an integer $1 \le \tau \le 2^n$ such that (16) holds.
- For each integer $1 \le j \le 2^n$, there exists a unique integer $1 \le k_j \le \tau$ such that $\delta_{2^n}^{j} \in \mathcal{R}_{k_j}^{\circ}(S)$, where $\mathcal{R}_k^{\circ}(S) = \mathcal{R}_k(S) \setminus \mathcal{R}_{k-1}(S)$, $\mathcal{R}_0(S) := \emptyset$.
- Let $1 \le p_j \le 2^m$ be such that $1 \le l \le 2^{m+n}$ and

$$\begin{cases} \delta_{2^n}^{i_l} \in S, & k_j = 1, \\ \delta_{2^n}^{i_l} \in \mathcal{R}_{k_j-1}(S), & 2 \le k_j \le \tau, \end{cases}$$

where $l = (p_j - 1)2^n + j$.

Corollary 1 (H. Li et al., Automatica, 2015)

The state feedback based output tracking control can be designed as $u(t) = Kx(t)$ with $K = \delta_{2^m}[p_1\ p_2 \cdots\ p_{2^n}]$.

Output Tracking of BNs

Output Tracking Criterion-Set Controllability Approach

Definition 5

Consider BCN (8) and given a nonempty set $S \subseteq \Delta_{2^n}$.
1. S is said to be a control invariant subset of BCN (8), if for any $x_0 \in S$, there exists a control $u_{x_0} \in \Delta_{2^m}$, such that $x(1; x_0, u_{x_0}) \in S$.
2. S is said to be globally reachable, if for any $x_0 \in \Delta_{2^n}$, there exist $T \in \mathbb{Z}_+$ and a control sequence $\{u(t) : t = 0, 1, \cdots, T-1\}$, such that $x(T; x_0, u) \in S$.

Output Tracking of BNs

Output Tracking Criterion-Set Controllability Approach

- Construct $P_0 = \left\{\{\delta_{2^n}^1\}, \cdots, \{\delta_{2^n}^{2^n}\}\right\}$.
- For $S \subseteq \Delta_{2^n}$ and $S \neq \emptyset$, construct

$$\bar{P}_0 = \{s_i^0 : i = 1, \cdots, |S|\},\ s_i^0 = \{x^i \in S\},\ P_d = \{S\}.$$

Theorem 5

The output of BCN (8) tracks the reference signal $y^* = \delta_{2^p}^{\alpha}$, if and only if there exists a nonempty set $S \subseteq \mathcal{O}(\alpha)$ satisfying the following two conditions:

(i). S is a control invariant subset of BCN (8) $\Leftrightarrow$ BCN (8) is one-step set controllable with respect to $\bar{P}_0$ and $P_d \Leftrightarrow J_d^\top M \bar{J}_0 > 0$, where J_d and $\bar{J}_0$ are the index matrices of P_d and $\bar{P}_0$, respectively;

(ii). S is globally reachable $\Leftrightarrow$ BCN (8) is set controllable with respect to P_0 and $P_d \Leftrightarrow J_d^\top \mathcal{C} J_0 > 0$, where J_0 is the index matrix of P_0.

Output Tracking of BNs

An Example

Example 2

Recall Example 1. Assume that the outputs are

$$\begin{cases} y_1(t) = x_1(t), \\ y_2(t) = x_2(t). \end{cases}$$

Our objective is to verify whether or not BCN (12) is trackable with respect to $Y^* = (1, 0)$ via set controllability approach.

- ◇ Using the vector form of Boolean variables, we have

$$H = \delta_4[1\ 1\ 2\ 2\ 3\ 3\ 4\ 4],\ y^* = \delta_4^2.$$

- ◇ According to (15), one can calculate that

$$\mathcal{O}(2) = \{\delta_8^3, \delta_8^4\}.$$

Output Tracking of BNs

An Example

- Construct

$$P_0 = \left\{\{\delta_8^1\}, \cdots, \{\delta_8^8\}\right\},\ \bar{P}_0 = P_d = \{\{\delta_8^3\}\}.$$

Correspondingly, we have $J_0 = \Delta_8$, $\bar{J}_0 = J_d = \delta_8^3$.

- On one hand, since

$$J_d^\top M \bar{J}_0 = 1 > 0,$$

BCN (12) is one-step set controllable with respect to $\bar{P}_0$ and P_d.

- On the other hand, since

$$J_d^\top \mathcal{C} J_0 > 0,$$

BCN (12) is set controllable with respect to P_0 and P_d.

- By virtue of Theorem 5, $S = \{\delta_8^3\} \subseteq \mathcal{O}(2)$ is a control invariant subset of BCN (12). In addition, S is globally reachable. Therefore, the output of BCN (12) tracks the reference signal $y^* = \delta_4^2$, which coincides with the conclusion obtained in Li et al. 2015.

Outline

Concluding Remarks

1. Set controllability is a powerful tool to deal with observability and output tracking of BNs.
2. Further study: (i) Generalization of set controllability; (ii) Applications to networked evolutionary games*, discrete event systems* and finite-field networks*; (iii) Computational complexity reduction; (iv) Sampled-data control, event-triggered control, pinning control*.

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Concluding Remarks

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Thanks!