Singular Logical Systems

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Singular Boolean networks

Background:

- It is well known that singular systems, which are also referred to as differential-algebraic equations, descriptor systems or implicit systems, are often much more convenient and natural than standard models in the description of many science and engineering systems.

- Singular Boolean networks (SBNs), are also called dynamic-algebraic Boolean networks, which are much more effective than standard Boolean networks for describing many science systems, due to the existence of algebraic constraints in the relations between states for practical scene.

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Description of singular Boolean networks

➢ Dynamic logical equations with static logical equations constraints

Consider the following Boolean network with n nodes [21], which has r dynamic logical equations and $n-r$ static logical equations:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \ldots, x_n(t)), \\ x_2(t+1) = f_2(x_1(t), x_2(t), \ldots, x_n(t)), \\ \qquad \vdots \\ x_r(t+1) = f_r(x_1(t), x_2(t), \ldots, x_n(t)), \\ \delta_2^{i_{r+1}} = f_{r+1}(x_1(t), x_2(t), \ldots, x_n(t)), \\ \qquad \vdots \\ \delta_2^{i_n} = f_n(x_1(t), x_2(t), \ldots, x_n(t)), \end{cases} \tag{1}$$

where $f_j(x_1(t), x_2(t), \ldots, x_n(t))$, $j = 1, 2, \ldots, n$ are logical functions of $x_1(t), x_2(t), \ldots, x_n(t)$, $i_j = 1$ or $i_j = 2$, $j = r+1, r+2, \ldots, n$.

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$$x_i(t+1) = f_i(x_1(t), \cdots, x_n(t)),$$

where $x_i \in \mathcal{D}_k$ are logical variables and $f_i : \mathcal{D}_k^n \to \mathcal{D}_k$, $i = 1, 2, \cdots, n$ are k-valued logical functions.

conventional Boolean networks

$r(r < n)$ dynamic logical equations

$n - r$ static logical equations

dynamic-algebraic Boolean networks

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Description of singular Boolean networks

In the $n-r$ static logical equations of (1), assume that $x_{r+1}, x_{r+2}, \ldots, x_n$ can be represented by $x_1, x_2, \ldots, x_r$; that is, there are logical functions $\bar{f}_j$ of logical variables $(x_1, x_2, \ldots, x_r)$ such that $x_j = \bar{f}_j(x_1, x_2, \ldots, x_r), j = r+1, r+2, \ldots, n$. Substituting them into (1), we have the equivalent form of (1):

$$\begin{cases} x_1(t+1) = \bar{f}_1(x_1(t), x_2(t), \ldots, x_r(t)), \\ x_2(t+1) = \bar{f}_2(x_1(t), x_2(t), \ldots, x_r(t)), \\ \quad\quad\vdots \\ x_r(t+1) = \bar{f}_r(x_1(t), x_2(t), \ldots, x_r(t)), \\ x_{r+1}(t) = \bar{f}_{r+1}(x_1(t), x_2(t), \ldots, x_r(t)), \\ \quad\quad\vdots \\ x_n(t) = \bar{f}_n(x_1(t), x_2(t), \ldots, x_r(t)), \end{cases} \quad (2)$$

where, for any j from 1 to r,

$$\bar{f}_j(x_1(t), \ldots, x_r(t)) = f_j(x_1(t), \ldots, x_r(t), \bar{f}_{r+1}(x_1(t), \ldots, x_r(t)), \ldots, \bar{f}_n(x_1(t), \ldots, x_r(t))).$$

Furthermore, given initial value $(x_1(0), x_2(0), \ldots, x_r(0))$ all logical variables $(x_1(t), x_2(t), \ldots, x_n(t))$ of (2) are determined at any time t.

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Description of singular Boolean networks

Example 1.

$$\begin{cases} A(t+1) = B(t) \wedge C(t), \\ B(t+1) = \neg A(t), \\ \delta_2^1 = (A(t) \rightarrow B(t)) \leftrightarrow C(t). \end{cases}$$

The dynamic and static relationships

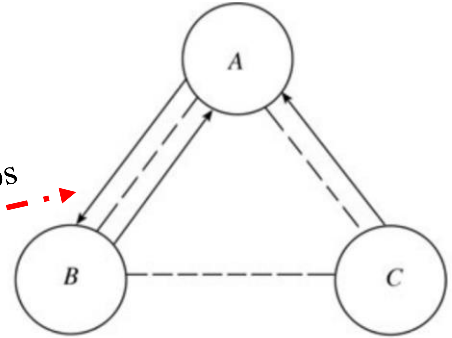

Figure 1 The solid line represents dynamic relationships; while the dash line represents static ones.

$$\begin{cases} A(t+1) = B(t) \wedge (A(t) \rightarrow B(t)), \\ B(t+1) = \neg A(t), \\ C(t) = A(t) \rightarrow B(t), \end{cases}$$

The dynamic and static relationships

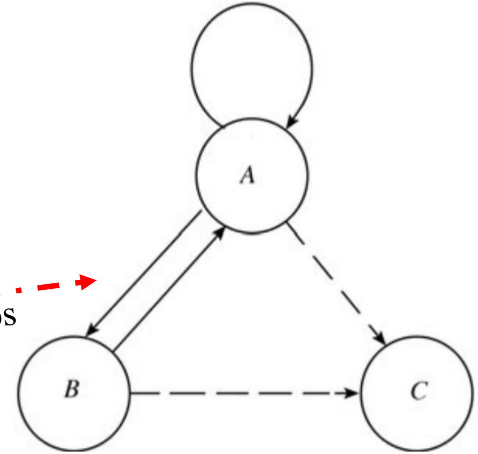

Figure 2 $A \rightarrow B$ means that $A(t)$ affects $B(t+1)$; $A \dashrightarrow C$ means that $A(t)$ affects $C(t)$ not $C(t+1)$.

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➤ Two kinds of algebraic form of (1)

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \ldots, x_n(t)), \\ x_2(t+1) = f_2(x_1(t), x_2(t), \ldots, x_n(t)), \\ \quad \vdots \\ x_r(t+1) = f_r(x_1(t), x_2(t), \ldots, x_n(t)), \\ \delta_2^{i_{r+1}} = f_{r+1}(x_1(t), x_2(t), \ldots, x_n(t)), \\ \quad \vdots \\ \delta_2^{i_n} = f_n(x_1(t), x_2(t), \ldots, x_n(t)), \end{cases}$$

STP

$$\begin{cases} x_1(t+1) = M_1 x(t), \\ x_2(t+1) = M_2 x(t), \\ \quad \vdots \\ x_r(t+1) = M_r x(t), \\ \delta_2^1 = M_{r+1} x(t), \\ \quad \vdots \\ \delta_2^1 = M_n x(t). \end{cases}$$

$$\begin{cases} x_1(t+1) = M_1 x(t), \\ x_2(t+1) = M_2 x(t), \\ \quad \vdots \\ x_r(t+1) = M_r x(t), \\ \delta_2[1,1] x_{r+1}(t+1) = M_{r+1} x(t), \\ \quad \vdots \\ \delta_2[1,1] x_n(t+1) = M_n x(t). \end{cases}$$

STP

another condensed algebraic expression of (1)

$$\check{E} x^1(t+1) = M x(t) \qquad (12)$$

$$x^1(t+1) = \ltimes_{i=1}^{r} x_i(t+1)$$

$$x(t) = \ltimes_{i=1}^{n} x_i(t)$$

$$\check{E} = I_{2^r} \otimes \delta_{2^{n-r}}^1 \in \mathcal{L}_{2^n \times 2^r}$$

STP

$$Ex(t+1) = Mx(t)$$

Matrix E is a singular one, the BN (9) (or its equivalent form (1)) is a singular BN

$$x(t) = \ltimes_{i=1}^{n} x_i(t)$$

$$M = M_1 * M_2 * \cdots * M_n$$

$$E = I_{2^r} \otimes \delta_{2^{n-r}}[1, 1, \ldots, 1]$$

Contents

Normalization of singular Boolean networks

- **Problem statement**

Normalization problem of singular Boolean networks

$$Ex(t+1) = Mx(t), \quad (13)$$

$x(t) = \ltimes_{i=1}^{n} x_i(t)$

$E, M \in \mathcal{L}_{2^n \times 2^n}$

$\text{rank}(E) = 2^r \leqslant 2^n$

If E is a nonsingular logical matrix,

$$x(t+1) = \widetilde{M}x(t) := E^T M x(t)$$

If E is singular logical matrix, when or how can $Ex(t+1) = Mx(t)$ be converted into a normal dynamic Boolean network with an algebraic constraint?

Definition 2: Consider the singular Boolean network (13) The normalization problem is solvable, if we can find a coordinate transformation z = T x such that under z coordinate frame, the singular BN (13) is equivalent to the following BN:

$$\begin{cases} z^1(t+1) = M^1 z^1(t), \\ z^2(t) = M^2 z^1(t), \end{cases} \quad (14)$$

where $z(t) = z^1(t) \ltimes z^2(t)$, $M^1 \in \mathcal{L}_{2^r \times 2^r}$ and $M^2 \in \mathcal{L}_{2^{n-r} \times 2^r}$.

$$Ex(t+1) = Mx(t)$$

$z = Tx$

$$\begin{cases} z^1(t+1) = \bar{M}^1 z(t), \\ \delta_{2^{n-r}}^1 = \bar{M}^2 z(t), \end{cases} \quad (15)$$

get $z^2(t) = M^2 z^1(t)$ from $\delta_{2^{n-r}}^1 = \bar{M}^2 z(t)$

$$\begin{cases} z^1(t+1) = M^1 z^1(t), \\ z^2(t) = M^2 z^1(t), \end{cases}$$

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Normalization of singular Boolean networks

- **Solvability of the normalization problem**

$$Ex(t+1)=Mx(t) \Rightarrow \begin{cases} z^1(t+1)=\bar{M}^1 z(t), \\ \delta^1_{2^{n-r}}=\bar{M}^2 z(t), \end{cases} \quad (15)$$

$$E=\begin{bmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,2^n} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,2^n} \\ \vdots & \vdots & & \vdots \\ e_{2^n,1} & e_{2^n,2} & \cdots & e_{2^n,2^n} \end{bmatrix}$$

Theorem 1. There exist nonsingular matrices P, Q all belonging to $\mathcal{L}_{2^n\times 2^n}$ such that $PEQ=I_{2^r}\otimes L$ with $L\in\mathcal{L}_{2^{n-r}\times 2^{n-r}}$ being a constant mapping matrix, if and only if, for any row of E, either $\sum_{j=1}^{2^n} e_{i,j}=0$ or $\sum_{j=1}^{2^n} e_{i,j}=2^{n-r}$.

Theorem 2. The solvability of converting (13) into (15) is equivalent to, for any row of matrix E, either.

$$\sum_{j=1}^{2^n} e_{i,j}=0 \text{ or } \sum_{j=1}^{2^n} e_{i,j}=2^{n-r}$$

Without loss of generality, we assume that $L=\delta_{2^{n-r}}[1,1,\dots,1]$

$$Ex(t+1)=Mx(t) \rightarrow PEx(t+1)=PMx(t) \xrightarrow{z=Q^{\mathrm{T}}x} PEQz(t+1)=PMQz(t).$$

$$\xrightarrow{PEQ=I_{2^r}\otimes L} PEQz(t+1)=z^1(t+1)\ltimes Lz^2(t+1)$$

$$\xrightarrow{PMQ=\bar{M}_1*\bar{M}_2} PMQz(t)=\bar{M}_1 z(t)\ltimes \bar{M}_2 z(t)$$

$$\Bigg\} \quad z^1(t+1)\ltimes\delta^1_{2^{n-r}}=\bar{M}_1 z(t)\ltimes\bar{M}_2 z(t)$$

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Normalization of singular Boolean networks

$$\begin{cases} z^1(t+1) = \bar{M}^1 z(t), \\ \delta_{2^{n-r}}^1 = \bar{M}^2 z(t), \end{cases} \quad \Rightarrow \quad \begin{cases} z^1(t+1) = M^1 z^1(t), \\ z^2(t) = M^2 z^1(t), \end{cases}$$

biconditional matrix

$$M_{\leftrightarrow}^k = \delta_k[1, \underbrace{k, k, \ldots, k}_{k}, 1, \ldots, 1, \underbrace{k, k, \ldots, k}_{k}, 1] \in \mathcal{L}_{k \times k^2}.$$

Theorem 3. $z^2(t)$ can be solved from $\delta_{2^{n-r}}^1 = \bar{M}^2 z(t)$, if $\bar{M}^2 = M_{\leftrightarrow} M^2$, where $M_{\leftrightarrow} \in \mathcal{L}_{2^{n-r} \times 2^{2(n-r)}}$ is the biconditional matrix, and $M^2 \in \mathcal{L}_{2^{n-r} \times 2^r}$.

Proof. If $\bar{M}^2 = M_{\leftrightarrow} M^2$, then we get $\delta_{2^{n-r}}^1 = M_{\leftrightarrow} M^2 z^1(t) z^2(t)$. Thus we have $z^2(t) = M^2 z^1(t)$.

this condition is not necessary

Corollary 1. Assuming matrix $\bar{M}^2 = \delta_{2^{n-r}}[i_1, i_2, \ldots, i_{2^n}] \in \mathcal{L}_{2^{n-r} \times 2^n}$, if there exist biconditional matrix $M_{\leftrightarrow} \in \mathcal{L}_{2^{n-r} \times 2^{2(n-r)}}$ and $M^2 \in \mathcal{L}_{2^{n-r} \times 2^r}$ such that $\bar{M}^2 = M_{\leftrightarrow} M^2$, then the numbers of 1 and 2^{n-r} in $\{i_1, i_2, \ldots, i_{2^n}\}$ are 2^{n-r} and $(2^n - 2^{n-r})$, respectively.

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Normalization of singular Boolean networks

single-conditional logical matrix set

$$M^k_{\leftrightharpoons} = \{\delta_k[1, \underbrace{i^1_1, i^1_2, \ldots, i^1_k}_{k}, 1, \ldots, 1, \underbrace{i^{k-1}_1, i^{k-1}_2, \ldots, i^{k-1}_k}, 1] \in \mathcal{L}_{k\times k^2}\}$$

with $i^l_j \neq 1, l = 1, 2, \ldots, k-1, j = 1, 2, \ldots, k.$

Theorem 4. $z^2(t)$ can be uniquely solved from $\delta^1_{2^{n-r}} = \bar{M}^2 z(t)$ if and only if $\bar{M}^2 = M_{\leftrightharpoons} M^2$, where $M_{\leftrightharpoons} \in M^{2^{n-r}}_{\leftrightharpoons} \subseteq \mathcal{L}_{2^{n-r}\times 2^{2(n-r)}}$ is a single-conditional matrix, and $M^2 \in \mathcal{L}_{2^{n-r}\times 2^r}$.

Proof. (Sufficiency) If $\bar{M}^2 = M_{\leftrightharpoons} M^2$, then we get $\delta^1_{2^{n-r}} = M_{\leftrightharpoons} M^2 z^1(t) z^2(t)$. Thus we have $z^2(t) = M^2 z^1(t)$.

(Necessity) Assume that $z^2(t)$ can be uniquely solved from $\delta^1_{2^{n-r}} = \bar{M}^2 z(t)$. Assume that $z^2(t) = M^2 z^1(t)$. Then $\delta^1_{2^{n-r}} = M_{\leftrightharpoons} M^2 z^1(t) z^2(t)$. Therefore, $\bar{M}^2 = M_{\leftrightharpoons} M^2$ is derived.

Corollary 2. Split matrix $\bar{M}^2 = \delta_{2^{n-r}}[i_1, i_2, \ldots, i_{2^n}] \in \mathcal{L}_{2^{n-r}\times 2^n}$ into k equal size parts, that is, $\bar{M}^2 = \delta_{2^{n-r}}[i_1, i_2, \ldots, i_k, | \ldots, | i_{2^n-k+1}, i_{2^n-k+2}, \ldots, i_{2^n}]$, then there exist a single-conditional matrix $M_{\leftrightharpoons} \in M^{2^{n-r}}_{\leftrightharpoons} \subseteq \mathcal{L}_{2^{n-r}\times 2^{2(n-r)}}$ and $M^2 \in \mathcal{L}_{2^{n-r}\times 2^r}$ such that $\bar{M}^2 = M_{\leftrightharpoons} M^2$, if and only if there is only one $\delta^1_{2^{n-r}}$ in every part.

$$\bar{M}^2 = \begin{bmatrix} \bar{m}^2_{1,1} & \bar{m}^2_{1,2} & \cdots & \bar{m}^2_{1,2^n} \\ \bar{m}^2_{2,1} & \bar{m}^2_{2,2} & \cdots & \bar{m}^2_{2,2^n} \\ \vdots & \vdots & & \vdots \\ \bar{m}^2_{2^{n-r},1} & \bar{m}^2_{2^{n-r},2} & \cdots & \bar{m}^2_{2^{n-r},2^n} \end{bmatrix}$$

Corollaries 1 and 2

⇓

$z^2(t)$ cannot be solved from $\delta^1_{2^{n-r}} = \bar{M}^2 z(t)$ if $\sum_{j=1}^{2^n} \bar{m}^2_{1,j} \neq 2^{n-r}$.

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Normalization of singular Boolean networks

- **The normalization of $\check{E}x^1(t+1) = Mx(t)$**

As for the normalization problem for the second form of singular Boolean network, that is,

$$\check{E}x^1(t+1) = Mx(t), \quad (25)$$

where $x^1(t+1) = \ltimes_{i=1}^{r} x_i(t+1), x(t) = \ltimes_{i=1}^{n} x_i(t)$, $\check{E} \in \mathcal{L}_{2^n \times 2^r}$, we have the following.

Theorem 5. Singular Boolean network (25) has equivalent form (15), if and only if $\check{E} \in \mathcal{L}_{2^n \times 2^r}$ is full of column rank.

Corollary 3. Singular Boolean network (25) has equivalent form (15), if and only if $\mathrm{Col}_i(\check{E}) \neq \mathrm{Col}_j(\check{E})$ if $i \neq j$.

Proof. The necessity is obvious. Here only give the proof of the sufficiency. Assume that $\check{E} \in \mathcal{L}_{2^n \times 2^r}$ is full of column ranks. Then via some row permutations, that is, with nonsingular matrix $P \in \mathcal{L}_{2^n \times 2^n}$, we obtain

$$P\check{E} = \delta_{2^n}[1, 2^{n-r}+1, 2 \cdot 2^{n-r}+1, \ldots, (2^r-1) \cdot 2^{n-r}+1].$$

It is easy to check $P\check{E} = I_{2^r} \otimes \delta_{2^{n-r}}^1$. Then left-multiplying matrix P in both sides of (25), we get $P\check{E}x^1(t+1) = PMx(t)$. For the left side of $P\check{E}x^1(t+1) = PMx(t)$, we have $P\check{E}x^1(t+1) = x^1(t+1) \ltimes \delta_{2^{n-r}}^1$. Additionally, for the right side, there exist matrices $\bar{M}_1 \in \mathcal{L}_{2^r \times 2^n}, \bar{M}_2 \in \mathcal{L}_{2^{n-r} \times 2^n}$ such that $PMx(t) = \bar{M}_1 x(t) \ltimes \bar{M}_2 x(t)$ with $PM = \bar{M}_1 * \bar{M}_2$. Thus we have $x^1(t+1) \ltimes \delta_{2^{n-r}}^1 = \bar{M}_1 x(t) \ltimes \bar{M}_2 x(t)$, which is equivalent to

$$\begin{cases} x^1(t+1) = \bar{M}^1 x(t), \\ \delta_{2^{n-r}}^1 = \bar{M}^2 x(t). \end{cases} \quad (26)$$

Eq. (26) is just same as (15).

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Solvability of singular Boolean networks

Consider solution problem about the following dynamic Boolean equation:

$$Gx(t+1) = Hx(t), \qquad (27)$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t)$, $G, H \in \mathcal{L}_{2^n \times 2^n}$. It is easy to see that, for any initial value $x(0)$, Eq. (27) has solution $x(t)$ if and only if $\text{rank}[G\ H] = \text{rank}[G]$. First we discuss the uniqueness of the solution. See the simple numerical example below.

Example Consider: $Gx(t+1) = Hx(t)$,

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- $rank[G\ H] = rank[G]$; for any initial value $x(0)$, $Gx(t+1) = Hx(t)$ has solution $x(t)$.
- When $x(0) = [1\ 0\ 0\ 0]^{\mathrm{T}}$ *or* $x(0) = [0\ 0\ 1\ 0]^{\mathrm{T}}$, the solution is not unique.
- When $x(0) = [0\ 0\ 0\ 1]^{\mathrm{T}}$, the solution is unique.
- When $x(0) = [0\ 1\ 0\ 0]^{\mathrm{T}}$, the solution in the first step is unique, while in the second step it is not.

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$$Gx(t+1) = Hx(t),$$

$$G = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,2^n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,2^n} \\ \vdots & \vdots & & \vdots \\ g_{2^n,1} & g_{2^n,2} & \cdots & g_{2^n,2^n} \end{bmatrix}, \quad H = \begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,2^n} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,2^n} \\ \vdots & \vdots & & \vdots \\ h_{2^n,1} & h_{2^n,2} & \cdots & h_{2^n,2^n} \end{bmatrix}$$

For the same row of G and H, take i th row.

- If $\sum_{j=1}^{2n} g_{ij} > 1$, $\sum_{j=1}^{2n} h_{ij} \neq 0$, then the solution $x(t+1)$ is not unique for initial value x(0) satisfying $Hx(0) = \delta_{2n}^{i}$.
- If $\sum_{j=1}^{2n} g_{ij} = 0$, $\sum_{j=1}^{2n} h_{ij} \neq 0$, then the Eq. does not have solution for initial value x(0) satisfying $Hx(0) = \delta_{2n}^{i}$.

Theorem 6. Singular Boolean equation (27) has a unique solution for any initial value $x(0)$, if and only if $\mathrm{rank}[G\ H] = \mathrm{rank}[G]$, and for any row of these two matrices, $\sum_{j=1}^{2^n} h_{i,j} \neq 0$ implying $\sum_{j=1}^{2^n} g_{i,j} = 1$.

Corollary 4. If singular Boolean networks with form (13) satisfies condition in Theorem 1, and $n \neq r$, then Eq. (13) does not have a unique solution $x(1)$ for any $x(0)$.

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Solvability of singular Boolean networks

If singular Boolean network system satisfies the condition in **Theorem 1** (or **Theorem 5**), and the condition in Theorem 4

$$\begin{aligned} Ex(t+1) &= Mx(t) \\ \check{E}x^1(t+1) &= Mx(t) \end{aligned} \quad \Longleftrightarrow \quad \begin{cases} z^1(t+1) = M^1 z^1(t), \\ z^2(t) = M^2 z^1(t). \end{cases} \quad (31)$$

Admissible initial values for singular BN

$$z(0) \text{ satisfies } z^2(0) = M^2 z^1(0)$$

Via coordinate transformation $x = Q\,z$
the corresponding solution x(t) of (13) is derived

If the condition of Theorem 4 does not hold, and only condition in Theorem 1 (or Theorem 5) is satisfied

$$\begin{aligned} Ex(t+1) &= Mx(t) \\ \check{E}x^1(t+1) &= Mx(t) \end{aligned} \quad \Longleftrightarrow \quad \begin{cases} z^1(t+1) = \bar{M}^1 z(t), \\ \delta^1_{2^{n-r}} = \bar{M}^2 z(t). \end{cases} \quad (32)$$

$$\{z(0) : \delta^1_{2^{n-r}} = \bar{M}^2 z(0)\}$$

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Proposition 1. Denote $\bar{M}^2 = \delta_{2^{n-r}}[i_1, i_2, \ldots, i_{2^n}]$. $\{z(0) : \delta_{2^{n-r}}^1 = \bar{M}^2 z(0)\} \neq \emptyset$ if and only if there exists at least one element in $\{i_1, i_2, \ldots, i_{2^n}\}$ equal to 1.

For any admissible initial value, Eq. (31) has a unique solution, but Eq. (32) has not. Let us see the following numerical example.

Example 8. Consider singular Boolean network (32) with the following coefficients:

$$\bar{M}^1 = \delta_4[1, 3, 3, 4, 2, 2, 3, 4], \quad \bar{M}^2 = \delta_2[1, 2, 2, 2, 2, 2, 1, 2].$$

Taking the admissible initial value $z(0) = \delta_8^7$ which satisfies $\delta_{2^{n-r}}^1 = \bar{M}^2 z(0)$

$z^1(1) = \delta_4^3$ there is no $z^2(1)$ satisfying $\delta_{2^{n-r}}^1 = \bar{M}^2 z^1(1) z^2(1)$

Taking the admissible initial value $z(0) = \delta_8^1$, we obtain $z(t) = \delta_8^1$ for any time t, which implies that δ_8^1 is the fixed point.

Corollary 5. For any admissible initial value, a singular Boolean network (32) has a unique solution, if and only if for any $z(0) \in \{z(0) : \delta_{2^{n-r}}^1 = \bar{M}^2 z(0)\}$ there exists a unique z^2 such that $\bar{M}^1 z(0) z^2 \in \{z(0) : \delta_{2^{n-r}}^1 = \bar{M}^2 z(0)\}$.

There exists a unique z^2 such that $\bar{M}^1 z(0) z^2 \in \{z(0) : \delta_{2^{n-r}}^1 = \bar{M}^2 z(0)\}$ is equivalent to the fact that z^2 can be uniquely solved from $\delta_{2^{n-r}}^1 = \bar{M}^2 z(t)$. Recalling Theorem 4, we derive the following.

Theorem 7. For any admissible initial value, singular Boolean network (32) has a unique solution, if and only if singular Boolean network (32) can be equivalently converted into form (31).

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Fixed points and cycles

In this section we discuss fixed points and cycles of singular Boolean networks with form (13), that is,

$$Ex(t+1) = Mx(t), \tag{33}$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t)$, $E, M \in \mathcal{L}_{2^n \times 2^n}$, and $\text{rank}(E) = 2^r \leqslant 2^n$. To this end, we first generalize definitions of fixed points and cycles [8] to the singular case.

Definition 5. 1) A state $x_0 \in \Delta_{2^n}$ is called a fixed point of singular Boolean network (33) if $Ex_0 = Mx_0$.

2) $\{x_0, x_1, \ldots, x_k\}$ is called a cycle of singular Boolean network (33) with length k if $x_k = x_0$ and the elements in the set $\{x_0, x_1, \ldots, x_{k-1}\}$ are pairwise distinct.

The following theorem shows how many fixed points a Boolean network has.

[8] Cheng D, et al. Analysis and Control of Boolean Networks: A Semi-Tensor Product Approach. 2011

Feng J, et al. Singular Boolean networks: Semi-tensor product approach[J], 2013

Theorem 8. Consider singular Boolean network (33). $\delta_{2^n}^i$ is its fixed point, if and only if $\mathrm{Col}_i(E) = \mathrm{Col}_i(M)$. It follows that the number of fixed point of singular Boolean network (33), denoted by N_e, equals the number of i for which $\mathrm{Col}_i(E) = \mathrm{Col}_i(M)$.

Proof. Assume that $\delta_{2^n}^i$ is the fixed point. Note that $E\delta_{2^n}^i = \mathrm{Col}_i(E)$ and $M\delta_{2^n}^i = \mathrm{Col}_i(M)$. It is clear that $\delta_{2^n}^i$ is the fixed point if and only if $\mathrm{Col}_i(E) = \mathrm{Col}_i(M)$.

If singular Boolean network (33) can be normalized, that is, it has equivalent form (14), then from the theorem above and section 5 of [8], the following result is derived directly.

Corollary 6. Assume singular Boolean network (33) can be normalized into (14). Then the number of i for which $\mathrm{Col}_i(E) = \mathrm{Col}_i(M)$ is equal to $\mathrm{tr}(M^1)$, where M^1 is same in (16). Additionally, if $\delta_{2^n}^i$ is a fixed point of (14), then $Q^{\mathrm{T}}\delta_{2^n}^i$ is the corresponding fixed point of (33).

with normalization

We can discuss cycles of singular Boolean network (33) indirectly if it has normalized form (14). We could first derive all cycles of (14) using method provided in Section 5 of [8], and then get those of (33). If $\{x_0, x_1, \ldots, x_k\}$ is a cycle of (14), then $\{Q^{\mathrm{T}}x_0, Q^{\mathrm{T}}x_1, \ldots, Q^{\mathrm{T}}x_k\}$ is one of (33). However, in so doing we need to normalize (33) first.

[8] Cheng D, et al. Analysis and Control of Boolean Networks: A Semi-Tensor Product Approach. 2011

Assume that $\text{row}(M) \subseteq \text{row}(E)$ in a singular Boolean network (33). Then there exists a nonsingular matrix $P \in \mathcal{L}_{2^n \times 2^n}$ such that $M = PE$. Denote $Ex(t) = y(t)$. Then Eq. (33) is equivalent to $y(t+1) = Py(t)$. Thus from Section 5 of [8], we get all cycles of $y(t+1) = Py(t)$. For one cycle $\{y_0, y_1, \ldots, y_k\}$ of $y(t+1) = Py(t)$, if $y_i \in \text{Col}(E), i = 0, 1, \ldots, k$, then we can derive the cycle $\{x_0, x_1, \ldots, x_k\}$ corresponding to $\{y_0, y_1, \ldots, y_k\}$. That means if there is some i such that y_i does not belong to $\text{Col}(E)$, then there is no cycle $\{x_0, x_1, \ldots, x_k\}$ corresponding to $\{y_0, y_1, \ldots, y_k\}$.

Furthermore, for 2 length cycles of singular Boolean networks we have the following result, which can be extended to k length cycle cases.

without normalization

Theorem 9. Consider singular Boolean network (33). $(\delta_{2^n}^i, \delta_{2^n}^j)$ is a cycle, if and only if $\text{Col}_i(E) = \text{Col}_j(M)$ and $\text{Col}_j(E) = \text{Col}_i(M)$. It follows that the number of cycles with length 2 of singular Boolean network (33), denoted by C_2, equals that number of pair (i, j) for which $\text{Col}_i(E) = \text{Col}_j(M)$ and $\text{Col}_j(E) = \text{Col}_i(M)$.

Proof. Assume that $(\delta_{2^n}^i, \delta_{2^n}^j)$ is a cycle. Note that $E\delta_{2^n}^j = \text{Col}_j(E)$ and $M\delta_{2^n}^i = \text{Col}_i(M)$. It is clear that $(\delta_{2^n}^i, \delta_{2^n}^j)$ is a cycle if and only if $\text{Col}_i(E) = \text{Col}_j(M)$ and $\text{Col}_j(E) = \text{Col}_i(M)$.

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Conclusion and discussion

$$Ex(t+1) = Mu(t)x(t)$$

$$\check{E}x^1(t+1) = Mu(t)x(t)$$

Two kinds of condensed algebraic forms

STP

The normalization problem

Some conditions for the solvability of the normalization problem

Solutions and admissible initial values

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Feng J, et al. Singular Boolean networks: Semi-tensor product approach[J], 2013

Disturbance decoupling of singular Boolean control networks

PRELIMINARIES

We study and analyze gene regulatory networks when some external disturbances exist, for example, in the healthy mechanisms of biological systems, cancer as usual be defined as failures, how to design a controller such that these external disturbances have no influence on the outputs of system is an important problem in biological systems, which is called ***disturbance decoupling problem*** (**DDP**).

Definition 1. *([39]) The STP of two matrices $A \in R_{m\times n}$ and $B \in R_{p\times q}$ is defined as*

$$A \ltimes B = (A \otimes I_{\frac{s}{n}})(B \otimes I_{\frac{s}{p}}). \quad (1)$$

Where $s = lcm(n, p)$ is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

In fact, conventional matrices product is a special situation of the STP, when $n = p$. Therefore, we usually call it "product", and omit the symbol "$\ltimes$".

Definition 2. *([40]) For $A \in M_{m\times r}$, $B \in M_{n\times r}$, the Khatri-Rao product of A and B is defined as*

$$A * B = [Col_1(A) \otimes Col_1(B), \cdots, Col_r(A) \otimes Col_r(B)].$$

Definition 3. *A matrix $A \in \mathcal{L}_{m\times n}$ is called a column-periodic matrix with period τ if τ is a proper factor of n such that $Col_{i+\tau}(A) = Col_i(A)$, $1 \leq i \leq n - \tau$.*

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Lemma 1. ([39]) Logical function $f(x_1, x_2, \ldots, x_n)$ with logical variables $x_1, x_2, \ldots, x_n \in \Delta_2$, and there is a unique matrix $M_f \in \mathcal{L}_{2\times 2^n}$, called the structure matrix of f, such that

$$f(x_1, x_2, \cdots, x_n) = M_f \ltimes_{i=1}^{n} x_i, \qquad (2)$$

where $\ltimes_{i=1}^{n} x_i = x_1 \ltimes x_2 \cdots \ltimes x_n.$

Lemma 2. ([30]) Given an integer r ≤ n, let M_G be the structure matrix of the given logical mapping $G = (g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)): D^n \rightarrow D^k$.Split M_G into 2^r blocks as $M_G = [(M_G)_1, (M_G)_2, \ldots, (M_G)_{2^r}]$ $x_{r+1}, \ldots, x_n$ are redundant variables if and only if

$$\text{rank}((M_G)_i) = 1, \quad \forall i = 1,\ 2, \cdots, 2^r, \qquad (3)$$

where $(M_G)_i \in \mathcal{L}_{2^k \times 2^{n-r}}$.

Definition 4. ([24]) Given an integer t ≥ 2, a t-type is a set of t logical matrices of dimension t × t. That is,

$$T = \{T_1, T_2, \cdots, T_t \mid T_i \in \mathcal{L}_{t\times t}, 1 \leq i \leq t\}.$$

[39] D. Cheng, et al. Analysis and control of Boolean networks: a semi-tensor product approach. 2010.

[30] M. Yang, et al. Controller design for disturbance decoupling of Boolean control networks, 2013.

Consider the following static equations

$$1 = k_i(x_1, \cdots, x_n), \quad i = r+1, \cdots, n, \tag{4}$$

where k_i: $\mathcal{D}_n \longrightarrow \mathcal{D}$ are logical function, and $x_i \in \mathcal{D}, i = 1, \cdots, n$. Let $k = (k_1, \cdots, k_r)$ and $X^1 = \ltimes_{i=1}^r x_i \in \Delta_p$ with $p = 2^r$, and $X^2 = \ltimes_{i=r+1}^n x_i \in \Delta_q$ with $q = 2^{n-r}$.

STP

$$\delta_q^1 = M_k X^1 X^2, \tag{5}$$

where M_k is the structure matrix of k.

$$\varphi_i = \{E_i \in \mathcal{L}_{q\times q} \mid Col_i(E_i) = \delta_q^1; Col_j(E_i) \neq \delta_q^1, j \neq i\}$$

where $i = 1, \cdots, q$. Then, one can construct a set of q-types as $\mathcal{E}_q \equiv \{\{E_1, \cdots, E_q\} \mid E_i \in \varphi_i, i = 1, \cdots, q\}$. (6)

Now, we consider when X^2 can be solved as functions of X^1, i.e., when (4) can be expressed as

$$x_i = k_i(x_1, \cdots, x_r), \quad i = r+1, \cdots, n. \tag{7}$$

Lemma 3. *((Implicit Function Theorem)[24]) Suppose the structure matrix of k associated with (4) can be expressed as $M_k = [M_1, \cdots, M_q]$. Then x_i $(i = r+1, \cdots, n)$ can be solved as (7) from (4) if and only if there exists a q-type $T = \{E_1, \cdots, E_q\} \in \mathcal{E}_q$, such that*

$$M_i \in T, \quad i = 1, \cdots, q. \tag{8}$$

[24] D. Cheng, et al. Bi-decomposition of multi-valued logical functions and its applications, 2013.

In general, we consider the following SBCN, which consists of n nodes, m control inputs, q' disturbance inputs and p' outputs (e ≤ r)

$$\begin{cases} x_i(t+1) = f_i(\bar{U}(t), \bar{X}(t), \bar{\Xi}(t)), & i = 1, \cdots, r, \\ \delta_2^1 = f_k'(\bar{X}(t)), & k = r+1, \cdots, n, \\ y_j(t) = h_j(x_1(t), \cdots, x_e(t)), & j = 1, \cdots, p', \end{cases} \tag{10}$$

STP

$$\begin{cases} X^1(t+1) = L^1 U(t) X(t) \xi(t), \\ \delta_{2^{n-r}}^1 = L^2 X(t), \\ Y(t) = H X^3(t), \end{cases} \tag{11}$$

If L^2 satisfies Lemma 3

$$\begin{cases} X^1(t+1) = L^1 U(t) X(t) \xi(t), \\ X^2(t) = L^3 X^1(t), \\ Y(t) = H X^3(t). \end{cases} \tag{12}$$

Remark 1. If e > r, the rest of e − r states can be expressed by $X^1(t)$ from (4) and Lemma 3, then $Y(t)$ is a function on $X^1(t)$.Hence, we only consider the case e ≤ r without loss of generalization

D. Cheng, at al. Bi-decomposition of multi-valued logical functions and its applications, 2013.

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Output feedback control

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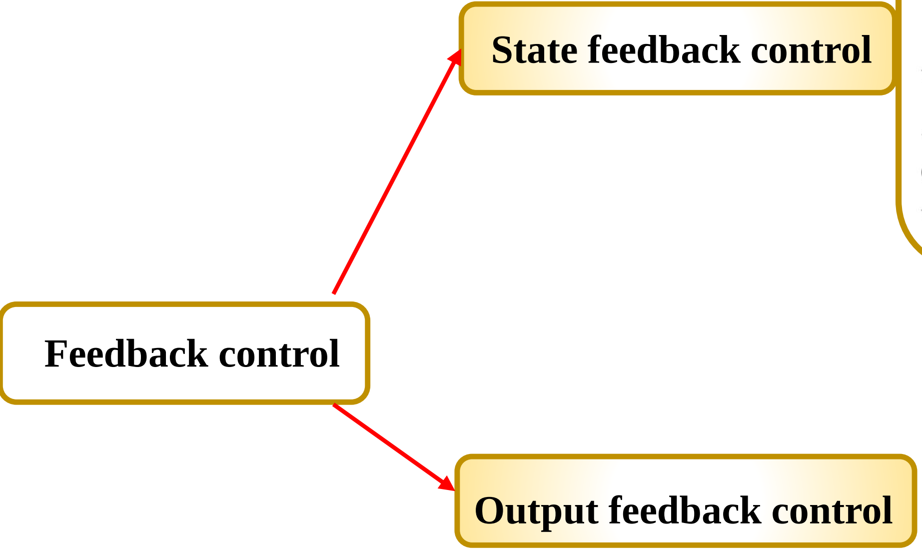
State feedback control



Considering system (12) with $e \leq r$, one can rewrite $X^3(t+1)$ by

$$X^3(t+1) = L^4 U(t) X(t) \xi(t), \tag{13}$$

where $L^4 = F_1 \Pi_{i=2}^{e} [(I_{2^{m+n+q'}} \otimes F_i) \Phi_{m+n+q'}] \in \mathcal{L}_{2^e \times 2^{m+n+q'}}$. We are now ready to design a state feedback controller in the form of $U(t) = KX(t)$, such that other state variables and the disturbances are independent on $X^3(t)$.

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State feedback control

$$X^3(t+1) = L^4 U(t) X(t) \xi(t)$$

$$L^4 = [(L^4)_1 \quad (L^4)_2 \cdots (L^4)_{2^m}]$$
$$(L^4)_j = [(L^4)_{j,1} \quad (L^4)_{j,2} \cdots (L^4)_{j,2^n}]$$
$$U(t) = KX(t)$$
$$K = \delta_{2^m}[v_1 \quad v_2 \cdots v_{2^n}]$$

$$\begin{aligned} X^3(t+1) &= L^4 K \Phi_n X^1(t) X^2(t) \xi(t) \\ &= L^4 K \Phi_n X^1(t) L^3 X^1(t) \xi(t) \\ &= L^4 K \Phi_n (I_{2^r} \otimes L^3) \Phi_r X^3(t) X^4(t) \xi(t) \\ &= (L^4)_v (I_{2^r} \otimes L^3) \Phi_r X^3(t) X^4(t) \xi(t), \end{aligned} \quad (14)$$

where $(L^4)_v = [(L^4)_{v_1,1} \quad (L^4)_{v_2,2} \cdots (L^4)_{v_{2^n},2^n}]$ and $X^4(t) = \ltimes_{i=e}^{r} x_i(t)$, $X^1(t) = X^3(t) X^4(t)$.

To simplify the expression, let $\Gamma = (I_{2^r} \otimes L^3)\Phi_r$ and $L^3 = \delta_{2^{n-r}}[v_1 \quad v_2 \cdots v_{2^r}]$, then we have

$$\Gamma = \begin{bmatrix} L^3 & 0_{2^{n-r} \times 2^r} & \cdots & 0_{2^{n-r} \times 2^r} \\ 0_{2^{n-r} \times 2^r} & L^3 & \cdots & 0_{2^{n-r} \times 2^r} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{2^{n-r} \times 2^r} & 0_{2^{n-r} \times 2^r} & \cdots & L^3 \end{bmatrix} \Phi_r$$
$$= \begin{bmatrix} \delta_{2^{n-r}}^{v_1} & 0_{2^{n-r}} & \cdots & 0_{2^{n-r}} \\ 0_{2^{n-r}} & \delta_{2^{n-r}}^{v_2} & \cdots & 0_{2^{n-r}} \\ \vdots & \vdots & \vdots & \vdots \\ 0_{2^{n-r}} & 0_{2^{n-r}} & \cdots & \delta_{2^{n-r}}^{v_{2^r}} \end{bmatrix}. \quad (15)$$

$\Gamma = \delta_{2^n}[m_1 \quad m_2 \cdots m_{2^r}]$, where $m_1 = v_1, m_2 = v_2 + 2^{n-r}, \cdots, m_{2^r} = v_{2^r} + 2^{n-r} \times (2^r - 1)$

define a set $\Lambda = \{m_1, m_2, \ldots, m_{2^r}\}$

$$\begin{aligned} X^3(t+1) &= (L^4)_0 \Phi_n \delta_{2^n}[m_1 \quad m_2 \cdots m_{2^r}] X^3 X^4 \xi(t) \\ &= (L^4)_v \delta_{2^n}[m_1 \quad m_2 \cdots m_{2^r}] X^3 X^4 \xi(t) \\ &= (L^4)_{v_m} X^3(t) X^4(t) \xi(t), \end{aligned} \quad (16)$$

where $(L^4)_0 = [(L^4)_{v_1} \quad (L^4)_{v_2} \cdots (L^4)_{v_{2^n}}]$ and $(L^4)_{v_m} = [(L^4)_{v_{m_1},m_1} \quad (L^4)_{v_{m_2},m_2} \cdots (L^4)_{v_{m_{2^r}},m_{2^r}}]$.

The DDP could be solved by $U(t) = KX(t)$, if and only if $X^4(t)$,$\xi(t)$ are redundant variables in (16).

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State feedback control

The DDP could be solved by $U(t) = KX(t)$, if and only if $X^4(t)$,$\xi(t)$ are redundant variables in (16).

$$L' = [(L')_1 \quad (L')_2 \cdots (L')_{2^e}],$$

where $(L')_\sigma = [(L^4)_{v_{m_{\sigma[1]}}} \quad (L^4)_{v_{m_{\sigma[2]}}} \cdots (L^4)_{v_{m_{\sigma[2^{r-e}]}}}] \in \mathcal{L}_{2^e \times 2^{q'+r-e}}$, $\forall \sigma \in \Omega_e$ and $\sigma[l] = (\sigma - 1)2^{r-e} + l$, for all $l \in \Omega_{r-e}$.

By Lemma 2, $X^4(t), \xi(t)$ are redundant variables if and only if

$$\mathrm{rank}((L')_\sigma) = 1, \quad \mathrm{for all} \sigma \in \Omega_e. \qquad (17)$$

Now, for any integer $\rho \in \Omega_r$, we define the following sets: $\Psi_{m_\rho} = \{k_\theta \in \Omega_e$: there exists an integer $\lambda \in \Omega_m$ such that $(L^4)_{\lambda, m_\rho} = \delta_{2^e}[k_\theta \cdots k_\theta]\}$;

$\mathcal{V}_{m_\rho}^{k_\theta} = \{\lambda \in \Omega_m : (L^4)_{\lambda, m_\rho} = \delta_{2^e}[k_\theta \cdots k_\theta]$, for all $k_\theta \in \Psi_{m_\rho}\}$.

Theorem 1. The DDP of SBN (12) is solved, if and only if

$$\Upsilon_\sigma := \bigcap_{l=1}^{2^{r-e}} \Psi_{m_l} \neq \emptyset, \quad \mathrm{for all} \sigma \in \Omega_e. \qquad (18)$$

Additionally, if (18) holds, then the state feedback matrix can be designed as $K = \delta_{2^m}[v_1 \quad v_2 \cdots v_{2^n}]$, *where* $v_{m_{\sigma[l]}} \in \mathcal{B}_{\sigma[l]} := \bigcup_{k_{\theta_i^\sigma} \in \Upsilon_\sigma} \mathcal{V}_{m_{\sigma[l]}}^{k_{\theta_i^\sigma}}$, *for all* $\sigma, i \in \Omega_e$, $l \in \Omega_{r-e}$, *and the rest of* v_ρ *are free.*

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Proof: (Sufficiency) Suppose

$$K = \delta_{2^m}[v_1 \quad v_2 \cdots v_{2^n}],$$

then one can get the state feedback control $U(t) = KX(t)$ that

$$\begin{aligned} X^3(t+1) =&(L^4)_v \delta_{2^n}[m_1, m_2, \cdots, m_{2^r}] X^3 X^4 \xi(t) \\ =&(L^4)_{v_m} X^3(t) X^4(t) \xi(t). \end{aligned} \tag{19}$$

We partition $[(L^4)_{v_{m_1},m_1} \quad (L^4)_{v_{m_2},m_2} \cdots (L^4)_{v_{m_{2^r}},m_{2^r}}]$ into 2^e parts as:

Part 1: $[(L^4)_{v_{m_1},m_1} \cdots (L^4)_{v_{m_{2^{r-e}}},m_{2^{r-e}}}]$,

Part 2: $[(L^4)_{v_{m_{2[1]}},m_{2[1]}} \cdots (L^4)_{v_{m_{2[2^{r-e}]}},m_{2[2^{r-e}]}}]$,

$$\vdots$$

Part 2^r: $[(L^4)_{v_{m_{2^e[1]}},m_{2^e[1]}} \cdots (L^4)_{v_{m_{2^e[2^{r-e}]}},m_{2^e[2^{r-e}]}}]$.

Since $\Upsilon_\sigma \neq \emptyset$, for all $\sigma \in \Omega_e$, then there exists at least a $\beta_i \in \Upsilon_\sigma$, for any $\sigma \in \Omega_e$ such that the rank of every part equals 1. Therefore, the DDP of SBN (12) is solved by Lemma (2) and (17).

(Necessity) If the DDP of SBN (12) is solved, we can get that the rank of every part equals 1 by Lemma (2) and (17), $\mathrm{rank}[(\mathrm{L}^4)_{\mathrm{v}_{\mathrm{m}_{\mathrm{i}[1]}},\mathrm{m}_{\mathrm{i}[1]}} \cdots (\mathrm{L}^4)_{\mathrm{v}_{\mathrm{m}_{\mathrm{i}[2^{r-e}]}},\mathrm{m}_{\mathrm{i}[2^{r-e}]}}] = 1$, $i = 1, \cdots, 2^e$, in other words, $\Upsilon_\sigma \neq \emptyset$, for all $\sigma \in \Omega_e$. Assume that there are j_i elements in Υ_σ, for any $i \in \Omega_e$, that is $\{k_{\theta_1^1}, k_{\theta_2^1}, \cdots, k_{\theta_{j_1}^1}\} = \Upsilon_1$, so does $\{k_{\theta_1^{2^e}}, k_{\theta_2^{2^e}}, \cdots, k_{\theta_{j_{2^e}}^{2^e}}\} = \Upsilon_{2^e}$. For any $l \in \Omega_{r-e}$, we define the following sets:

$$\mathcal{B}_{1[l]} = \bigcup_{k_{\theta_i^1} \in \Upsilon_1} \mathcal{V}_{m_{1[l]}}^{k_{\theta_i^1}},$$

$$\mathcal{B}_{2[l]} = \bigcup_{k_{\theta_i^2} \in \Upsilon_2} \mathcal{V}_{m_{2[l]}}^{k_{\theta_i^2}},$$

$$\vdots$$

$$\mathcal{B}_{2^e[l]} = \bigcup_{k_{\theta_i^{2^e}} \in \Upsilon_{2^e}} \mathcal{V}_{m_{2^e[l]}}^{k_{\theta_i^{2^e}}}.$$

Therefore, we can get $v_{m_{\sigma[l]}} \in \mathcal{B}_{\sigma[l]}$ by (16). Actually, from the above analysis, one can find the rest elements of K have nothing to do with the DDP due to the existence of Γ.

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Contents

State feedback control

Output feedback control

Output feedback control

Based on the result obtained in **Theorem 1**, we analyze how to design **output feedback controller** in the form of

$$u_i(t) = \mathscr{J}_i(y_1(t), \cdots, y_{p'}(t)), \quad i = 1, \cdots, m, \qquad (20)$$

STP

$$U(t) = RY(t), \qquad (21)$$

$$\ltimes_{i=1}^{e} x_i(t) = x_1(t) \cdots x_e(t) \cdot 1_{2^{n-e}} x_{e+1}(t) \cdots x_n(t) = (I_{2^e} \otimes 1_{2^{n-e}}) X(t).$$

$$Y(t) = HX^3(t) = H(I_{2^e} \otimes 1_{2^{n-e}})X(t) = K'X(t)$$

For for all $k \in \Omega_{p'}$, we denote $\mathcal{O}(k)$ by the set of states whose output is $\delta_{2^{p'}}^{k}$.

$$\mathcal{O}(k) = \{\delta_{2^n}^{i} : Col_i(K') = \delta_{2^{p'}}^{k}\}.$$

$$\mathcal{R}(k) = \begin{cases} \bigcap\limits_{\delta_{2^n}^{i} \in \mathcal{O}(k) \cap \Lambda} \mathcal{B}_i, & \text{if } \mathcal{O}(\mathrm{k}) \cap \Lambda \neq \emptyset, \\ \{1, 2, \cdots, 2^m\}, & \text{if } \mathcal{O}(\mathrm{k}) \cap \Lambda = \emptyset \ \text{ or } \ \mathcal{O}(\mathrm{k}) = \emptyset. \end{cases} \qquad (23)$$

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Theorem 2. Assume that (18) holds. The DDP of SBCN (12) is solved by the output feedback control if and only if

$$\mathcal{R}(k) \neq \emptyset, \quad \text{forallk} \in \Omega_{p'}. \qquad (24)$$

Moreover, if (24) holds, then we can obtain all the output feedback gain matrices

$$R = \delta_{2^m}[\alpha_1, \alpha_2, \cdots, \alpha_{2^{p'}}], \quad \alpha_k \in \mathcal{R}(k). \qquad (25)$$

Proof: (Sufficiency) Suppose that (24) holds, and the output feedback gain matrices are in the form of (25). Assuming $K' = \delta_{2^{p'}}[k_1 \quad k_2 \cdots k_{2^n}] \in L_{2^{p'} \times 2^n}$, we have

$$RK' = [\alpha_{k_1}, \alpha_{k_2}, \cdots, \alpha_{k_{2^n}}]. \qquad (26)$$

Since $\alpha_{k_i} \in \mathcal{R}(k_i) \subseteq \mathcal{B}_i, \forall i \in \Omega_n$,from Theorem 1 it is learnt that the DDP is solvable by the state feedback control $U(t) = (\mathcal{R}K')X(t)$. Therefore, the DDP is solved by the output feedback control $U(t) = \mathcal{R}Y(t)$.

(Necessity) If (24) does not hold, there exists an integer $k' \in \Omega_{p'}$ such that $\mathcal{R}(k') = \emptyset$, i.e.$\mathcal{O}(k') \neq \emptyset$ and$\bigcap_{\delta_{2^n}^i \in \mathcal{O}(k') \cap \Lambda} \mathcal{B}_i = \emptyset$, then we suppose that $\mathcal{O}(k') = \{\delta_{2^n}^{i_1}, \dots, \delta_{2^n}^{i_\epsilon}\}$ .Hence, $k_{i_1} = \cdots = k_{i_\epsilon} = k'$and we have $\alpha_{k_{i_1}} = \cdots = \alpha_{k_{i_\epsilon}} = \alpha_{k'}$ . .

Since the DDP is solvable by $U(t) = RY(t)$, one can obtain that $U(t) = (RK')X(t)$.is a state feedback controller. Therefore,$a_{k'} \in \bigcap_{\delta_{2^n}^i \in \mathcal{O}(k') \cap \Lambda} \mathcal{B}_i$, which is a contradiction to $\bigcap_{\delta_{2^n}^i \in \mathcal{O}(k') \cap \Lambda} \mathcal{B}_i = \emptyset$.

Consequently,$\mathcal{R}(k) \neq \emptyset$ for all $k \in \Omega_{p'}$.

Function perturbations on singular Boolean networks

- The expression of SBNS
- The transition matrix under function perturbations
- Local uniqueness of solutions to SBNS
- The topological structure under dynamic equations perturbations

The expression of SBNS

An SBN is a set of nodes, $x_1, x_2, \ldots, x_n$, in which r (r < n) nodes satisfy the following dynamic logical equations:

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \ldots, x_n(t)), \\ x_2(t+1) = f_2(x_1(t), x_2(t), \ldots, x_n(t)), \\ \quad \vdots \\ x_r(t+1) = f_r(x_1(t), x_2(t), \ldots, x_n(t)), \end{cases} \tag{1a}$$

for t ≥ 0 and the states of remainder $n - r$ nodes satisfy the following algebraic logical equations at t ≥ 0,

$$\begin{cases} 1 = f_{r+1}(x_1(t), x_2(t), \ldots, x_n(t)), \\ 1 = f_{r+2}(x_1(t), x_2(t), \ldots, x_n(t)), \\ \quad \vdots \\ 1 = f_n(x_1(t), x_2(t), \ldots, x_n(t)), \end{cases} \tag{1b}$$

$$\delta_2^1 = f_{r+1}(x_1(t), x_2(t), \ldots, x_n(t)) \wedge \cdots \wedge f_n(x_1(t), x_2(t), \ldots, x_n(t)).$$

$$\delta_2^1 = g(x_1(t), \ldots, x_n(t))$$

Rewrite system (1) in a simple way:

$$\begin{cases} x_i(t+1) = f_i(x_1(t), \ldots, x_n(t)), \ i \in [1, r], & \text{(a)} \\ \delta_2^1 = g(x_1(t), \ldots, x_n(t)), & \text{(b)} \\ g(x_1(t+1), \ldots, x_n(t+1)) = \delta_2^1, & \text{(c)} \end{cases} \tag{2}$$

STP

$$\begin{cases} x_i(t+1) = M_{f_i} x(t), i \in [1, r], & \text{(a)} \\ \delta_2^1 = M_g x(t), & \text{(b)} \\ M_g x(t+1) = \delta_2^1, & \text{(c)} \end{cases} \tag{3}$$

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The expression of SBNS

- Multiplying the equations in (3)(a) together yields

$$(I_{2^r} \otimes \mathbf{1}_{2^{n-r}})x(t+1) = L_f x(t), \tag{4a}$$

where $L_f = M_{f_1} \prod_{i=2}^{r} [(I_{2^n} \otimes M_{f_i})\Phi_n]$, and $\Phi_n = \mathrm{Diag}\{\delta_{2^n}^1, \delta_{2^n}^2, \ldots, \delta_{2^n}^{2^n}\}$ is the power reducing matrix,

- Multiplying (4a) from the left by $((I_{2^r} \otimes \mathbf{1}_{2^{n-r}})x(t+1))^T$ yields

$$1 = x^T(t+1)(I_{2^r} \otimes \mathbf{1}_{2^{n-r}})^T L_f x(t) = x^T(t+1) L_0 x(t),$$

$$\begin{aligned} L_0 &= (I_{2^r} \otimes \mathbf{1}_{2^{n-r}})^T L_f \\ &= \left[\delta_{2^n}^{(i_1-1)2^{n-r}+1} + \delta_{2^n}^{(i_1-1)2^{n-r}+2} + \cdots + \delta_{2^n}^{i_1 2^{n-r}}, \ldots, \delta_{2^n}^{(i_{2^n}-1)2^{n-r}+1} + \delta_{2^n}^{(i_{2^n}-1)2^{n-r}+2} + \cdots + \delta_{2^n}^{i_{2^n} 2^{n-r}}\right]. \end{aligned}$$

x(t + 1) still takes value arbitrarily from the set $\{\delta_{2^n}^{(i_j-1)2^{n-r}+1}, \ldots, \delta_{2^n}^{i_j 2^{n-r}}\}$ with $x(t) = \delta_{2^n}^j$,which implies that L_0 can be regarded as the transition matrix of (4a).

- Multiplying (3)(b), (3)(c) from the left and right by $(\delta_2^1)^T$, respectively, one obtains

$$\begin{cases} 1 = \mathrm{Row}_1(M_g)x(t), & \text{(a)} \\ \mathrm{Row}_1(M_g)x(t+1) = 1. & \text{(b)} \end{cases} \tag{4b}$$

- multiplying (4a), (4b)(a) and (4b)(b) together, system (3) can be rewritten by

$$Ex(t+1) = Lx(t), \tag{5}$$

The expression of SBNS

- Multiplying (5) from the left by $(Ex(t+1))^T$ yields

$$1 = x^T(t+1)L_c x(t), \quad (6)$$

where $L_c = E^T L$, which is the transition matrix of system (5).

From the above equation, the relation between L_c and L_0 can also be obtained as follows:

$$\begin{aligned} L_c = \; & ((I_{2^n} \otimes \text{Row}_1(M_g))\Phi_n)^T (I_{2^r} \otimes \mathbf{1}_{2^{n-r}})^T \\ & L_f (I_{2^n} \otimes \text{Row}_1(M_g))\Phi_n = \Gamma^T L_0 \Gamma, \end{aligned} \quad (7)$$

where $\Gamma = (I_{2^n} \otimes \text{Row}_1(M_g))\Phi_n$.

the solution set of static equation (3)(b). Let $\mathcal{N}$ be the solution set of (3)(b), called admissible state set. We assume $\mathcal{N} = \{\delta_{2^n}^{a_1}, \delta_{2^n}^{a_2}, \ldots, \delta_{2^n}^{a_h}\}$ $(a_1 < a_2 < \cdots < a_h)$. Let $\mathcal{S} = \Delta_{2^n} \setminus \mathcal{N}$, $\mathcal{S} = \{\delta_{2^n}^{b_1}, \delta_{2^n}^{b_2}, \ldots, \delta_{2^n}^{b_s}\}$ $\cdots < b_s)$. Clearly, $h + s = 2^n$, $\text{Col}_{a_i}(M_g) = \delta_2^1, a_i \in \{a_1, \ldots, a_h\}$ and $\text{Col}_{b_j}(M_g) = \delta_2^2, b_j \in \{b_1, \ldots, b_s\}$. Then, we have

$$\text{Row}_1(M_g) = [1 \cdots 1 \underset{b_1}{0} 1 \cdots \underset{\cdots}{} 1 \underset{b_s}{0} 1 \cdots 1],$$

$$\begin{aligned} \Gamma &= (I_{2^n} \otimes \text{Row}_1(M_g))\Phi_n \\ &= [\delta_{2^n}^1 \; \delta_{2^n}^2 \cdots \delta_{2^n}^{b_1-1} \underset{b_1}{\mathbf{0}_{2^n}^T} \; \delta_{2^n}^{b_1+1} \cdots \delta_{2^n}^{b_l-1} \underset{b_l}{\mathbf{0}_{2^n}^T} \; \delta_{2^n}^{b_l+1} \cdots \delta_{2^n}^{2^n}]. \end{aligned}$$

Matrix L_c can be obtained from L_0 by substituting the elements in the rows and columns with indexes $b_1, b_2, \ldots b_l$ by zeros.

The transition matrix under function perturbations

Definition 2. A one-function perturbation of SBN (3) is obtained if structure matrix M_{f_i} of a function f_i ($i \in [1, n]$) alters by changing the value on some $Col_j M_{f_i}$ for $j \in [1, n]$) ,that is, changing δ_2^1 to δ_2^2 or changing δ_2^2 to δ_2^1

$$L_f \xrightarrow{\text{function perturbations}} L'_f$$

$$L'_f = \delta_{2^r}[i'_1, \ldots, i'_{2^n}].$$

$$\mathrm{Col}_j(L_f) = \ltimes_{p=1}^{r} \mathrm{Col}_j(M_{f_p}), j \in [1, 2^n].$$

$\mathrm{Col}_j(M_{f_p})$ ($p \in [1, r]$) changes from $\delta_2^{i_j^p}$ to $\delta_2^{i'^p_j}$, $i_j^p, i'^p_j \in [1, 2]$

$\mathrm{Col}_j(L_f)$ changes from $\delta_{2^r}^{i_j}$ to $\delta_{2^r}^{i'_j}$.

$i_j = \sum_{p=1}^{r-1}(i_j^p - 1)2^{r-p} + i_j^r$, and $i'_j = i_j + (i'^k_j - i^k_j)2^{r-k}$.

Lemma 3. *Consider system (3) with its algebraic form (4). If the j-th ($j \in \Omega_n$) column of M_{f_k} ($k \in [1, r]$) alters, then $i_v = i'_v$, $v \in [1, 2^n] \setminus \{j\}$, and $L_f = \delta_{2^r}[i_1, \ldots, i_j, \ldots, i_{2^n}]$ becomes $L'_f = \delta_{2^r}[i_1, \ldots, i'_j, \ldots, i_{2^n}]$.*

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Suppose that there are m ($m \in [1, 2^n]$) columns of L_f changing under function perturbations, which are denoted k_1-th, $\cdots$, k_m- th. For simplicity, let $m = 2$, and assume that $\mathrm{Col}_{k_1}(L_f) = \delta_{2^r}^{\iota_{k_1}}$ and $\mathrm{Col}_{k_2}(L_f) = \delta_{2^r}^{\iota_{k_2}}$ change to $\delta_{2^r}^{\iota'_{k_1}}, \delta_{2^r}^{\iota'_{k_2}}$, respectively. Then, $L'_f = \overline{\Psi}_{k_1,k_2} + L_f \Psi_{k_1,k_2}$,

where $\Psi_{k_1,k_2} = [\delta_{2^n}^1 \ \delta_{2^n}^2 \ \cdots \ \delta_{2^n}^{k_1-1} \ \mathbf{0}_{2^n}^T \ \delta_{2^n}^{k_1+1} \cdots \ \delta_{2^n}^{k_2-1} \mathbf{0}_{2^n}^T \delta_{2^n}^{k_2+1} \cdots \delta_{2^n}^{2^n}] \in \mathcal{B}_{2^n \times 2^n}$, and $\overline{\Psi}_{k_1,k_2} = [\mathbf{0}_{2^r}^T \cdots \ \mathbf{0}_{2^r}^T \ \underset{k_1}{\delta_{2^r}^{i'_{k_1}}} \ \mathbf{0}_{2^r}^T \cdots \mathbf{0}_{2^r}^T \underset{k_2}{\delta_{2^r}^{i'_{k_2}}} \mathbf{0}_{2^r}^T \cdots \mathbf{0}_{2^r}^T]$

Denote matrix L'_c by the new L_c under perturbations, then it derives from (7) that

$$\begin{aligned} L'_c &= ((I_{2^n} \otimes \mathrm{Row}_1(M_g))\Phi_n)^T (I_{2^r} \otimes \mathbf{1}_{2^{n-r}})^T \\ &\quad L'_f (I_{2^n} \otimes \mathrm{Row}_1(M_g))\Phi_n \\ &= \Gamma^T (I_{2^r} \otimes \mathbf{1}_{2^{n-r}})^T \overline{\Psi}_{k_1,k_2} \Gamma + \Gamma^T L_0 \Psi_{k_1,k_2} \Gamma. \end{aligned} \tag{8}$$

Theorem 4. *$L'_c = L_c$, if $k_\beta \in \{b_1, \ldots, b_l\}$, for any $\beta \in [1, 2]$.*

Proof. If $k_\beta \in \{b_1, \ldots, b_l\}$ for any $\beta \in [1, 2]$, and the existence of Γ is to make L_0 substituted zeros in the rows and columns with in- dexes $b_1, \ldots, b_l$. It is easy to obtain that $\Gamma^T (I_{2^r} \otimes \mathbf{1}_{2^{n-r}})^T \overline{\Psi}_{k_1,k_2} \Gamma = \mathbf{0}_{2^n \times 2^n}$ and $\Gamma^T L_0 \Psi_{k_1,k_2} \Gamma = \Gamma^T L_0 \Gamma$. Then $L'_c = \mathbf{0}_{2^n \times 2^n} + \Gamma^T L_0 \Gamma = L_c$ from (8), which completes the proof.

Now, we consider function perturbations occur in static equa-tion (2)(b), which makes $\mathcal{N}$ change, assume that $\mathcal{N}' = \{\delta_{2^n}^{a'_1}, \ldots, \delta_{2^n}^{a'_{h'}}\}$ ($a'_1 < \cdots < a'_{h'}$) It is learned from (4a) that $L'_f = L_f$.

Assume that $\mathcal{N}' \cap \mathcal{N} := \{\delta_{2^n}^{\hat{a}_1}, \ldots, \delta_{2^n}^{\hat{a}_{\hat{h}}}\}$, and $\mathcal{N}' \setminus \mathcal{N} := \{\delta_{2^n}^{\bar{a}_1}, \ldots, \delta_{2^n}^{\bar{a}_w}\}(w + \hat{h} = h')$.

Denote Γ under static equation perturbations by Γ'. $\Gamma' = \Lambda_{\bar{a}_1,\ldots,\bar{a}_w} + \Gamma \Lambda_{\hat{a}_1,\ldots,\hat{a}_{\hat{h}}}$,

where $\Lambda_{\hat{a}_1,\ldots,\hat{a}_{\hat{h}}} = [\mathbf{0}_{2^n}^T \cdots \ \mathbf{0}_{2^n}^T \ \underset{\hat{a}_1}{\delta_{2^n}^{\hat{a}_1}} \ \mathbf{0}_{2^n}^T \cdots \ \mathbf{0}_{2^n}^T \ \underset{\hat{a}_2}{\delta_{2^n}^{\hat{a}_2}} \ \mathbf{0}_{2^n}^T \cdots \mathbf{0}_{2^n}^T \ \underset{\hat{a}_{\hat{h}}}{\delta_{2^n}^{\hat{a}_h}}]$

and $\Lambda_{\bar{a}_1,\ldots,\bar{a}_w} = [\mathbf{0}_{2^n}^T \cdots \ \mathbf{0}_{2^n}^T \ \underset{\bar{a}_1}{\delta_{2^n}^{\bar{a}_1}} \ \mathbf{0}_{2^n}^T \cdots \ \mathbf{0}_{2^n}^T \ \underset{\bar{a}_2}{\delta_{2^n}^{\bar{a}_2}} \ \mathbf{0}_{2^n}^T \cdots \ \mathbf{0}_{2^n}^T \ \underset{\bar{a}_w}{\delta_{2^n}^{\bar{a}_w}}]$.

Therefore, $L'_c = \Gamma'^T L_f \Gamma'$.

Local uniqueness of solutions to SBNS

Definition 5. The solution to SBN (3) is locally unique in terms of $\mathcal{W} \subseteq \mathcal{N}$, if for any $x(t) \in \mathcal{W}$, $x(t+1) \in \mathcal{W}$ can be uniquely determined by (6).

Actually, when $\mathcal{W} = \mathcal{N}$, the solution to SBN (3) is unique. Assume that $\mathcal{W} = \{\delta_{2^n}^{z_1}, \delta_{2^n}^{z_2}, \ldots, \delta_{2^n}^{z_s}\}$ $(z_1 < z_2 < \cdots < z_s)$, and denote Υ by the set of $\{z_1, \ldots, z_s\}$. From Definition 5, it is easy to obtain that $\Upsilon \subseteq \{a_1, \ldots, a_h\}$ and $s \leq h$.

Lemma 6. *The solution to SBN (3) is locally unique in terms of $\mathcal{W}$ if and only if*

$$Col_{z_\lambda}(L_c) \in \mathcal{W}, \quad \text{for } \lambda \in [1, s].$$

Proof. Sufficiency is trivial. As for the necessity, since the solution to SBN (3) is locally unique in terms of $\mathcal{W}$, then $x(t+1) \in \mathcal{W}$ can be uniquely determined by $1 = x^T(t+1)L_c x(t)$. Therefore, for all $\lambda \in [1, s]$, one can obtain that $\mathrm{Col}_{z_\lambda}(L_c) \in \mathcal{W}$.

Now, we define a matrix $I_{\{h_1,h_2;o_1,o_2\}}$, where $\mathrm{Col}_{o_1}(I_{2^n})$ and $\mathrm{Col}_{o_2}(I_{2^n})$ are substituted by $\mathrm{Col}_{h_1}(I_{2^n})$ and $\mathrm{Col}_{h_2}(I_{2^n})$, respectively.

Proposition 1. *Assume that the k_1-th and k_2-th columns of L_f $(k_1, k_2 \in \Upsilon)$ change under function perturbations, and $L_f' = \delta_{2^r}[i_1, \ldots, i_{k_1}', \ldots, i_{k_2}', \ldots, i_{2^n}]$ from Lemma 3. If there exist $q_1, q_2 \in \Upsilon \setminus \{k_1, k_2\}$ such that $i_{k_\beta}' = i_{q_\beta}$, $\beta \in [1, 2]$, then $L_c' = L_c I_{\{q_1,q_2;k_1,k_2\}}$.*

Proof. There exist $q_1, q_2 \in \Upsilon \setminus \{k_1, k_2\}$ such that $i_{k_\beta}' = i_{q_\beta}$, $\beta \in [1, 2]$, then one can get

$$L_f' = L_f I_{\{q_1,q_2;k_1,k_2\}}. \tag{10}$$

As a result, it follows from (8), (10) that

$$\begin{aligned} L_c' &= \Gamma^T L_0 I_{\{q_1,q_2;k_1,k_2\}} \Gamma \\ &= \Gamma^T L_0 \Gamma I_{\{q_1,q_2;k_1,k_2\}} = L_c I_{\{q_1,q_2;k_1,k_2\}}, \end{aligned} \tag{11}$$

which completes the proof.

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Local uniqueness of solutions to SBNS

Theorem 7. *Assume that the solution to SBN (3) is locally unique in terms of $\mathcal{W}$. If $L_c' = L_c I_{\{q_1,q_2;k_1,k_2\}}$ after function perturbations, where q_1, q_2, k_1, k_2 are given by Proposition 1, then the solution to system (9) is also locally unique under perturbations.*

Proof. The solution to SBN (3) is locally unique in terms of $\mathcal{W}$, which implies that $\mathrm{Col}_{z_\lambda}(L_c) \in \mathcal{W}$, for $\lambda \in [1, s]$. It follows from $L_c' = L_c I_{\{q_1,q_2;k_1,k_2\}}$ that $\mathrm{Col}_{z_\lambda}(L_c') \in \mathcal{W}$, for $\lambda \in [1, s]$ as well. Therefore, the solution to system (9) is locally unique under perturbations.

Remark 2. $L_c' = L_c I_{\{q_1,q_2;k_1,k_2\}}$ is a sufficient but not necessary condition for the invariance of local uniqueness, which can be shown by the following example.

Example 3. Consider the following SBN with its algebraic form as follows,

$$\begin{cases} (I_{2^2} \otimes 1_2)x(t+1) = L_f x(t), \\ \delta_2^1 = M_g x(t), \\ M_g x(t+1) = \delta_2^1, \end{cases} \tag{15}$$

where $L_f = \delta_{2^2}[2\ 4\ 3\ 4\ 1\ 3\ 3\ 3]$ and $M_g = \delta_2[1\ 2\ 1\ 1\ 2\ 1\ 1\ 2]$, which means $\mathcal{N} = \{\delta_{2^3}^1, \delta_{2^3}^3, \delta_{2^3}^4, \delta_{2^3}^6, \delta_{2^3}^7\}$. Therefore, the solution to SBN (15) is not unique by Meng and Feng (2014b). But if $\mathcal{W} = \{\delta_{2^3}^3, \delta_{2^3}^4, \delta_{2^3}^6, \delta_{2^3}^7\}$, it is learned from Lemma 6 that the solution to SBN (15) is locally unique in terms of $\mathcal{W}$.

If $\mathrm{Col}_4(L_f)$ changes from $\delta_{2^2}^4$ to $\delta_{2^2}^3$ and $\mathrm{Col}_6(L_f)$ changes from $\delta_{2^2}^3$ to $\delta_{2^2}^4$, then $L_f' = \delta_{2^2}[2\ 4\ 3\ 3\ 1\ 4\ 3\ 3]$. The solution to system (15) with function perturbation is locally unique as well by Lemma 6, but there does not exist $q_2 \in \{3, 4, 6, 7\} \setminus \{4, 6\}$ such that $i_6' = i_{q_2} = 4$. Therefore, $L_c' = L_c I_{\{q_1,q_2;k_1,k_2\}}$ is not a necessary condition for the invariance of local uniqueness.

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The topological structure under static equation perturbations

In this subsection, suppose that function perturbations only occur in static equation, which makes $\mathcal{W}$ change possibly, then assume that $\mathrm{Col}_{k_1}(M_g)$, $\mathrm{Col}_{k_2}(M_g)$, ..., $\mathrm{Col}_{k_{m'}}(M_g)$ change with $m' \leq 2^n$, $k_i \in [1, 2^n]$, $i \in [1, m']$. Then the new admissible state set $\mathcal{N}'$ is created. We define the new admissible state subset $\mathcal{W}' = \mathcal{N}' \cap \mathcal{W}$ and assume that $\mathcal{W}' = \{\delta_{2^n}^{z_1'}, \ldots, \delta_{2^n}^{z_{s'}'}\}$ $(z_1' < z_2' < \cdots < z_{s'}')$.

Definition 8. Assume that the solution to SBN (3) is locally unique in terms of $\mathcal{W}$. (i) A state $\delta_{2^n}^p$ in $\mathcal{W}$ is called a fixed point of SBN (3), if $1 = (\delta_{2^n}^p)^T L_c \delta_{2^n}^p$. (ii) $\{\delta_{2^n}^{g_0}, \ldots, \delta_{2^n}^{g_l}\} \subseteq \mathcal{W}$ is called a cycle of SBN (3) with length l, if $\delta_{2^n}^{g_l} = \delta_{2^n}^{g_0}$, $1 = (\delta_{2^n}^{g_{i+1}})^T L_c \delta_{2^n}^{g_i}$, $i \in [0, l-1]$, and elements in the set $\{\delta_{2^n}^{g_0}, \ldots, \delta_{2^n}^{g_{l-1}}\}$ are pairwise distinct.

Definition 9. The fixed point (or cycle) in $\mathcal{W}$ is stable if it is still a fixed point (or cycle) in $\mathcal{W}'$ after function perturbations, otherwise it is called an unstable fixed point (or cycle).

Theorem 10. *Assume that $\delta_{2^n}^p$ is a fixed point of system (3), then it is stable under static equation perturbations, if and only if for any $j \in [1, s']$, $Col_{z_j'}(L_c') \in \mathcal{W}'$, and $Col_p(L_c') = \delta_{2^n}^p$.*

Proof (***Sufficiency***). For any $j \in [1, s']$, $\mathrm{Col}_{z_j'}(L_c') \in \mathcal{W}'$, then the local uniqueness of solution is invariant under perturbations from Lemma 6. Moreover, $\mathrm{Col}_p(L_c') = \delta_{2^n}^p$ implies that $\delta_{2^n}^p \in \mathcal{W}'$ is also a fixed point of system (3). Therefore, fixed point $\delta_{2^n}^p$ is stable.

(Necessity): Since $\delta_{2^n}^p$ is a stable fixed point, then the local uniqueness of solutions to system (3) with function perturbations still holds, and $\delta_{2^n}^p \in \mathcal{W}'$ is also a fixed point. Therefore, it follows from Lemma 6 that for any $j \in [1, s']$, $\mathrm{Col}_{z_j'}(L_c') \in \mathcal{W}'$. Moreover, one can get $\mathrm{Col}_p(L_c') = \delta_{2^n}^p$ by Definition 8.

(Sufficiency): For any $j \in [1, s']$, since $\mathrm{Col}_{z_j'}(L_c') \in \mathcal{W}'$, then the local uniqueness of solution is invariant under perturbations. Moreover, if $\mathrm{Col}_{g_i}(L_c)$ is invariant, for $i \in [0, l-1]$, then $\mathrm{Col}_{g_i}(L_c') = \delta_{2^n}^{g_{i+1}}$, and as a result, the cycle is stable.

Theorem 11. *Assume that $\{\delta_{2^n}^{g_0}, \ldots, \delta_{2^n}^{g_l}\}$ is a cycle of system (3). When function perturbations happen in the static equation, then the cycle is stable if and only if for any $j \in [1, s']$, $Col_{z_j'}(L_c') \in \mathcal{W}'$, and $\mathrm{Col}_{g_i}(L_c)$ is invariant under perturbations for all $i \in [0, l-1]$.*

The necessity is similar with that in Theorem 10

The topological structure under dynamic equations perturbations

In this subsection, we consider the topological structure of SBN (3) as function perturbations only occur in dynamic equations, which implies that the admissible state set $\mathcal{N}' = \mathcal{N}$ and subset $\mathcal{W}' = \mathcal{W}$ in the following. Suppose that $\text{Col}_{k_\beta}(L_f) = \delta_{2^r}^{i_{k_\beta}}$ changes to $\delta_{2^r}^{i'_{k_\beta}}$, $\beta \in [1, m']$, and L_f changes to L'_f.

Lemma 12. *Assume that $\delta_{2^n}^p$ is a fixed point of system (3), then $\delta_{2^n}^p$ is a stable fixed point under function perturbations of dynamic equations, if and only if one of the following two cases hold:*

- *Case I. For any $\beta \in [1, m']$, $k_\beta \notin \Upsilon$.*
- *Case II. For any $\beta \in [1, m']$, $k_\beta \neq p$. Moreover, if there exists $\beta \in [1, m']$, such that $k_\beta \in \Upsilon$, then $\text{Col}_{k_\beta}(L'_c) \in \mathcal{W}$.*

Proof (***Sufficiency***). If for any $\beta \in [1, m']$, $k_\beta \notin \Upsilon$, then $\text{Col}_i(L_c)$, $i \in \Upsilon$ is invariant under perturbations. If case *II* holds, the local uniqueness of solution to system (3) with function perturbations is guaranteed, and the invariance of $\text{Col}_p(L_c)$ under perturbations. Thus $(\delta_{2^n}^p)^T L_c \delta_{2^n}^p = (\delta_{2^n}^p)^T L'_c \delta_{2^n}^p = 1$, which implies that fixed point $\delta_{2^n}^p$ is stable.

(Necessity): Since $\delta_{2^n}^p$ is a stable fixed point, then one can obtain that the local uniqueness of system (3) is invariant when the function perturbations occur in dynamic equations. Therefore, we can easily get Case *I* or Case *II*.

Theorem 13. *Assume that $\delta_{2^n}^p$ is the only fixed point of system (3), and the function perturbations occur in dynamic equations, then the new fixed point will be generated if and only if the following two conditions hold:*

- *Condition I. If there exists $\beta \in [1, m']$, such that $k_\beta \in \Upsilon$, then $\text{Col}_{k_\beta}(L'_c) \in \mathcal{W}$.*
- *Condition II. There exist $\beta \in [1, m']$, and $\alpha \in [1, 2^{n-r}]$ such that $(L'_c)_{((i'_{k_\beta}-1)2^{n-r}+\alpha,(i'_{k_\beta}-1)2^{n-r}+\alpha)} = 1$ with $(i'_{k_\beta} - 1)2^{n-r} + \alpha \neq p$.*

Proof. The proof of necessity is similar with that in Theorem 10, so we omit it here.

(Sufficiency): If there exists $\beta \in [1, m']$, such that $k_\beta \in \Upsilon$, then $\text{Col}_{k_\beta}(L'_c) \in \mathcal{W}$, which shows the local uniqueness of solutions is invariant under perturbations. It is learned from Definition 8 that if Condition *II* holds, then a new fixed point is generated.

The topological structure under dynamic equations perturbations

Lemma 14. *Suppose that $\{\delta_{2^n}^{g_0}, \ldots, \delta_{2^n}^{g_l}\}$ is a cycle of system (3), and the function perturbations occur in dynamic equations. The cycle is stable if and only if one of the following cases hold:*

- *Case I. For any $\beta \in [1, m']$, $k_\beta \notin \Upsilon$.*
- *Case II. For any $\beta \in [1, m']$, $k_\beta \notin \Xi$. Moreover, if there exists $\beta \in [1, m']$, such that $k_\beta \in \Upsilon$, then $\mathrm{Col}_{k_\beta}(L'_c) \in \mathcal{W}$.*

Proof. The proof of necessity is similar with that in Lemma 12, so we omit it here.

(Sufficiency): Assume that for any $\beta \in [1, m']$, $k_\beta \notin \Upsilon$, then $\mathrm{Col}_{g_i}(L_c)$ is invariant under perturbations. If Case II holds, the local uniqueness of solutions to system (3) with function perturbations is invariant, then one can get that $\mathrm{Col}_{g_i}(L_c)$ is also invariant under perturbations. Thus $(\delta_{2^n}^{g_{i+1}})^T L_c \delta_{2^n}^{g_i} = (\delta_{2^n}^{g_{i+1}})^T L'_c \delta_{2^n}^{g_i}$, $i \in [0, l-1]$, which implies that the cycle is stable.

Now, we consider one kind of unstable situations called shrink. In the following, we discuss two special cases. Suppose that $\{\delta_{2^n}^{g_0}, \delta_{2^n}^{g_1}, \ldots, \delta_{2^n}^{g_{l-1}}, \delta_{2^n}^{g_l}\}$ is a cycle of SBN (3) (see Fig. 1), and the local uniqueness of solution is invariant under perturbations. If there exists $\beta \in [1, m']$ such that $k_\beta = g_v$, and $1 = (\delta_{2^n}^{g_i})^T (L'_c) \delta_{2^n}^{g_v}$, $g_i, g_v \in \Xi$. Then there exist two cases: $i < v$ and $i > v$. When $i < v$, assume $\{\delta_{2^n}^{g_i}, \delta_{2^n}^{g_{i+1}}, \ldots, \delta_{2^n}^{g_{v-1}}, \delta_{2^n}^{g_v}\} \subset \mathcal{Q}$, and for any $\lambda \in \{g_i, g_{i+1}, \ldots, g_{v-1}, g_v\} \setminus \{g_v\}$, $\mathrm{Col}_\lambda(L'_c) = \mathrm{Col}_\lambda(L_c)$, then a shrunken cycle of SBN (3) is obtained under function perturbations, which is of length $v + 1 - i$ as shown in Fig. 2.

On the other hand, when $i > v$, assume $\{\delta_{2^n}^{g_0}, \delta_{2^n}^{g_1}, \ldots, \delta_{2^n}^{g_{v-1}}, \delta_{2^n}^{g_v}, \delta_{2^n}^{g_i}, \delta_{2^n}^{g_{i+1}}, \ldots, \delta_{2^n}^{g_{l-1}}\} \subset \mathcal{Q}$, and for any $\lambda \in \{g_0, g_1, \ldots, g_{v-1}, g_v, g_i, g_{i+1}, \ldots, g_{l-1}\} \setminus \{g_v\}$, $\mathrm{Col}_\lambda(L'_c) = \mathrm{Col}_\lambda(L_c)$, then a shrunken cycle of SBN (3) is obtained under function perturbations with length $v + l + 1 - i$ as shown in Fig. 3. In general, we have the following result.

The topological structure under dynamic equations perturbations

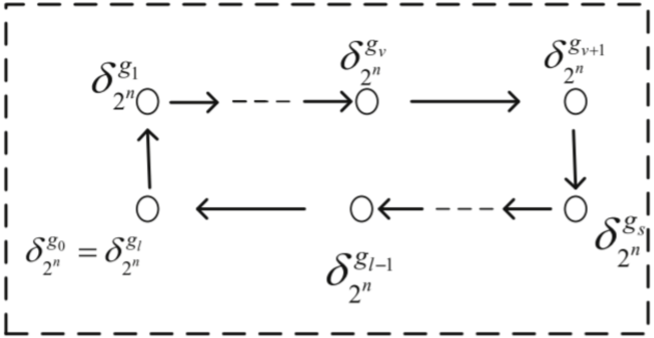

Fig. 1. A cycle of SBN (3).

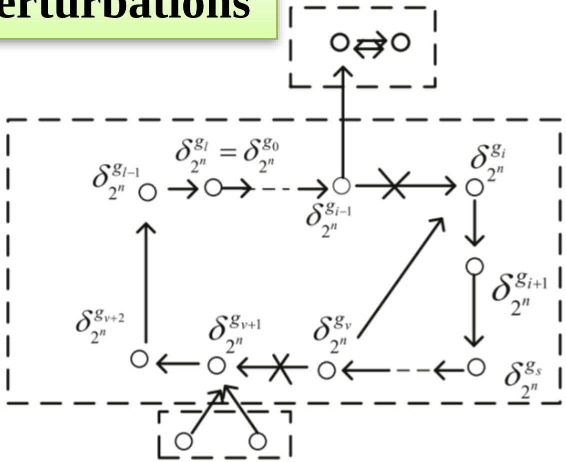

Fig. 2. A shrunken cycle of SBN (3) under function perturbations with $i < v$.

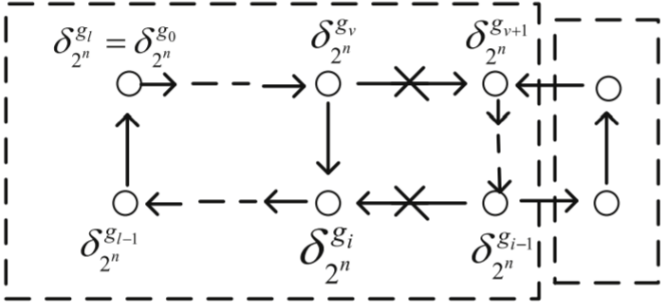

Fig. 3. A shrunken cycle of SBN (3) under function perturbations with $i > v$.

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The topological structure under static equation perturbations

Theorem 15. *Suppose that* $\{\delta_{2^n}^{g_0}, \dots, \delta_{2^n}^{g_l}\}$ *is a cycle of SBN* (3)*. The cycle shrinks if and only if the following two conditions hold:*

- *Condition I. If there exists* $\beta \in [1, m']$*, such that* $k_\beta \in \Upsilon$*, then* $Col_{k_\beta}(L_c') \in \mathcal{W}$.
- *Condition II. There exists* $\mathcal{Q}' \subset \mathcal{Q}$*, assumed by* $\{\delta_{2^n}^{g_{l_0}}, \delta_{2^n}^{g_{l_1}}, \dots, \delta_{2^n}^{g_{l_{z-1}}}\}$ *with* $\delta_{2^n}^{g_{l_z}} := Col_{g_{l_{z-1}}}(L_c) = \delta_{2^n}^{g_{l_0}}$, $l_0 < l_1 < \cdots < l_{z-1}$, $1 \leq z < l$*, such that* $\{\delta_{2^n}^{g_{l_0}}, \delta_{2^n}^{g_{l_1}}, \dots, \delta_{2^n}^{g_{l_z}}\}$ *is a new cycle of system* (3) *with function perturbations.*

Proof. Necessity is trivial. As for the sufficiency, if Condition I holds, then the local uniqueness of solution is invariant under perturbations. By Definition 8, if Condition II holds, a new cycle is generated, which implies that the cycle shrinks.

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Normalization and Solvability of Dynamic-Algebraic Boolean Networks

An **DABN** is a set of nodes $x_1, x_2, \ldots, x_n$, in which r (r < n) nodes satisfy the following dynamic logical equations:

Dynamic-Algebraic BNs

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), \ldots, x_n(t)), \\ x_2(t+1) = f_2(x_1(t), x_2(t), \ldots, x_n(t)), \\ \vdots \\ x_r(t+1) = f_r(x_1(t), x_2(t), \ldots, x_n(t)), \end{cases} \quad (1a)$$

for $t \geq 0$ and the states of remainder $n - r$ nodes satisfy the following algebraic logical equations at $t \geq 0$,

$$\begin{cases} 1 = f_{r+1}(x_1(t), x_2(t), \ldots, x_n(t)), \\ 1 = f_{r+2}(x_1(t), x_2(t), \ldots, x_n(t)), \\ \vdots \\ 1 = f_n(x_1(t), x_2(t), \ldots, x_n(t)), \end{cases} \quad (1b)$$

⇕ STP

$$\begin{cases} x^1(t+1) = Lx^1(t)x^2(t) & (2a) \\ \delta_{2^{n-r}}^1 = Gx^1(t)x^2(t) & (2b) \end{cases}$$

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Definition 2 [22]: If $\{x_{r+1}, \ldots, x_n\}$ can be expressed as functions of $\{x_1, \ldots, x_r\}$ by (2b), then DABNs can be transformed into standard BNs by substituting the variables $\{x_{r+1}, \ldots, x_n\}$ into (2a). This process is called the normalization of DABNs.

$$G = [G_1 \ G_2 \ \cdots \ G_{2^r}]$$

where $G_i \in \mathcal{L}_{2^{n-r} \times 2^{n-r}}, \ i \in Q_r$. Denote by $\mathcal{I}^1$ the set of indices i such that $\delta^1_{2^{n-r}} \in \mathrm{Col}(G_i)$. Let $\mathcal{X}^1$ be the set of all canonical vectors $\delta^i_{2^r}$ with $i \in \mathcal{I}^1$. Based on $\mathcal{X}^1$, we define $\mathcal{C}^1 = \Delta_{2^r} \backslash \mathcal{X}^1$ as the set of all canonical vector $\delta^i_{2^r}$, and then no vector $\delta^j_{2^{n-r}}$ can be found in $\mathcal{C}^1$ such that by assuming $x^1 = \delta^i_{2^r}$ and $x^2 = \delta^j_{2^{n-r}}$, condition (2b) holds. Moreover, denote by $\mathcal{I}^1_0$ the set of indices i such that only one of the columns of G_i coincides with $\delta^1_{2^{n-r}}$ and let $\mathcal{X}^1_0$ be the set of canonical vectors $\delta^i_{2^r}$ with $i \in \mathcal{I}^1_0$. It is obvious that $\mathcal{X}^1_0 \subset \mathcal{X}^1$.

Proposition 1: Given $x^1(t) = \delta^i_{2^r}$, a solution $x^2(t)$ to (2b) corresponding to $x^1(t)$ exists if and only if $i \in \mathcal{I}^1$, and is unique if and only if $i \in \mathcal{I}^1_0$.

[22] D. Cheng et al. Bi-decomposition of multi-valued logical functions and its applications, 2013.

Define a new matrix M_g of dimension $2^{n-r} \times 2^r$ with its entries being

$$\begin{cases} (M_g)_{ji} = 1, \text{ if } \mathrm{Col}_j(G_i) = \delta^1_{2^{n-r}} & \text{(3a)} \\ (M_g)_{ji} = 0, \text{ if } \mathrm{Col}_j(G_i) \neq \delta^1_{2^{n-r}}. & \text{(3b)} \end{cases}$$

if $x^1(t) \in \mathcal{X}^1$,
and $x^2(t)$ satisfies $(x^2(t))^T M_g x^1(t) \equiv 1$ (4)
then condition (2b) is satisfied.
if $x^1(t) \in \mathcal{C}^1$, then condition (2b) can never be satisfied.

For given $i \in \mathcal{I}^1$, denote the set of all j such that $(M_g)_{ji} = 1$ by $s(i)$. If $\sum_{j=1}^{2^{n-r}} (M_g)_{ji} > 1$, then there are more than one element in $s(i)$. In fact, for any $j \in s(i)$, $x^2(t) = \delta^j_{2^{n-r}}$ is the solution to (2b) corresponding to $x^1(t) = \delta^i_{2^r}$. Therefore, (2b) has multiple solutions x^2 when $\sum_{j=1}^{2^{n-r}} (M_g)_{ji} > 1$. Assuming $S(i) = \{\delta^j_{2^{n-r}} : j \in s(i)\}$, then we define the set of matrices

$$S(M_g) = \left\{ M : \mathrm{Col}_i(M) \in S(i) \text{ if } \sum_{j=1}^{2^{n-r}} (M_g)_{ji} > 1 \right. \\ \left. \text{otherwise } \mathrm{Col}_i(M) = \mathrm{Col}_i(M_g), i \in \mathcal{Q}_r \right\}. \quad (5)$$

Then (2*b*) can be equivalently rewritten as

$$x^1(t) \in \mathcal{X}^1 \text{ and } x^2(t) \in S(M_g)x^1(t). \quad (6)$$

Thus, DABN (2) is rewritten as

$$\begin{cases} x^1(t+1) = Lx^1(t)x^2(t) \\ x^2(t) \in S(M_g)x^1(t), \quad x^1(t) \in \mathcal{X}^1 \end{cases} \quad (7)$$

that is

$$x^1(t+1) \in \mathrm{L}x^1(t)S(M_g)x^1(t) = \mathrm{L}S(\bar{M}_g)x^1(t), \quad x^1(t) \in \mathcal{X}^1$$

where $S(\bar{M}_g) = (I_{2^r} \otimes S(M_g))\Phi_r$, (8)

when $\mathcal{I}^1 = \mathcal{I}^1_0$: $x^1(t+1) = L\bar{M}_g x^1(t), \quad x^1(t) \in \mathcal{X}^1$ (9)

where $\bar{M}_g = (I_{2^r} \otimes M_g)\Phi_r$.

Solvability of DABNs

Preliminaries

Let $\mathcal{X}$ be the solution set of (2b), i.e., the admissible state set of system (2). Since (6) is equivalent to (2b), we have

$$\mathcal{X} = \{x : \ x = x^1 x^2 \in S(\bar{M}_g) x^1, \quad x^1 \in \mathcal{X}^1\}.$$

(In particular, if $\mathcal{I}^1 = \mathcal{I}_0^1$, then $\mathcal{X} = \{x : \ x = x^1 M_g x^1, \ x^1 \in \mathcal{X}^1\}$.) Let $\mathcal{C} = \Delta_{2^n} \backslash \mathcal{X}$ and define $\mathcal{X}_0 = \{x : \ x = x^1 x^2 \in x^1 S(M_g) x^1, \ x^1 \in \mathcal{X}_0^1\}$. If the admissible state set $\mathcal{X} = \emptyset$, then DABN (2) is obviously unsolvable. Hence, we assume that $\mathcal{X} \neq \emptyset$ for a given DABN.

Definition 3: 1) Given an initial state $x(0) = x_0 \in \mathcal{X}$ and $x(t) \in \mathcal{X}, t > 0$. If both $x(0)$ and $x(t)$ satisfy system (2), then $x(t)$ is called a solution to system (2) with initial state $x(0)$.

2) Given an initial state $x(0) = x_0 \in \mathcal{X}$, DABN (2) is called solvable [the solution to DABN (2) is called unique] for the initial state $x(0)$, if there exists a solution (unique solution) $x(t), t > 0$, to system (2) with initial state $x(0)$.

3) DABN (2) is called solvable [the solution to DABN (2) is called unique], if it is solvable [the solution to DABN (2) is unique] for any initial state $x(0) \in \mathcal{X}$.

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Solvability of DABNs

Theorem 1: DABN (2) is solvable if and only if

$$LS(\bar{M}_g)x^1 \subset \mathcal{X}^1, \ \forall x^1 \in \mathcal{X}^1.$$

Proof: (Necessity) If DABN (2) is solvable, then for any initial state $x(0)$, we have $x(t) \in \mathcal{X}, t > 0$, which further means $x^1(t) \in \mathcal{X}^1$ from the definition of $\mathcal{X}$ and $\mathcal{X}^1$. Therefore, it follows from (8) that for any $x^1 \in \mathcal{X}^1$, $LS(\bar{M}_g)x^1 \subset \mathcal{X}^1$.

(Sufficiency) If for any $x^1(t-1) \in \mathcal{X}^1$, one has $LS(\bar{M}_g)x^1(t-1) \subset \mathcal{X}^1$ holds, which means $x^1(t) \in \mathcal{X}^1, t > 0$ by (8), then $x(t) \in \mathcal{X}$ using the definition of $\mathcal{X}$. Therefore, for any initial state $x(0) \in \mathcal{X}, x^1(0) \in \mathcal{X}^1$. As a result, $x(t) \in \mathcal{X}, t > 0$. Hence, DABN (2) is solvable. ■

Theorem 2: The solution to DABN (2) is unique if and only if

$$LS(\bar{M}_g)x^1 \subset \mathcal{X}_0^1 \ \forall x^1 \in \mathcal{X}^1.$$

$x^1(t+1) \in \mathrm{L}x^1(t)S(M_g)x^1(t) = \mathrm{L}S(\bar{M}_g)x^1(t), \ \ x^1(t) \in \mathcal{X}^1$ plays an important role in the way to find lower dimensional conditions for the solvability of DABNs

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Solvability of DABNs

Remark 4: Since $\mathcal{X}_0^1$ is not necessarily equivalent to $\mathcal{X}^1$, Theorem 2 shows that the uniqueness of solution to the system can be guaranteed even if the solution $x^2(t)$ to (2b) is not unique. The following example also shows the point.

$$\begin{cases} x_1(t+1) = x_2(t) \wedge x_3(t) \\ x_2(t+1) = \neg x_1(t) \\ 1 = (x_1(t) \wedge \neg x_3(t)) \vee (\neg x_1(t) \wedge \neg(x_2(t) \leftrightarrow x_3(t))). \end{cases}$$

Then $L = \delta_4[2, 4, 4, 4, 1, 3, 3, 3]$, and $G = \delta_2[2, 2, 1, 1, 2, 1, 1, 2]$. It is easy to see that $\mathcal{X}^1 = \{\delta_4^2, \delta_4^3, \delta_4^4\}$, and $\mathcal{X}_0^1 = \{\delta_4^3, \delta_4^4\} \neq \mathcal{X}^1$. Then we can obtain $S(M_g) = \{\delta_2[0, 1, 2, 1], \delta_2[0, 2, 2, 1]\}$ and $S(\bar{M}_g) = \{\delta_8[0, 3, 6, 7], \delta_8[0, 4, 6, 7]\}$. It is clear that for any $x_1 \in \mathcal{X}^1$, $L\delta_8[0, 3, 6, 7]x_1 \in \mathcal{X}_0^1$, and $L\delta_8[0, 4, 6, 7]x_1 \in \mathcal{X}_0^1$. Then the solution of the example is unique by Theorem 2.

Extension to DABCNs

In this section, we study the solution to DABCNs. Consider the dynamic system as

$$\begin{cases} x_1(t+1) = \tilde{f}_1(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t)) \\ x_2(t+1) = \tilde{f}_2(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t)) \\ \vdots \\ x_r(t+1) = \tilde{f}_r(u_1(t), \ldots, u_m(t), x_1(t), \ldots, x_n(t)) \end{cases} \quad (15)$$

STP

where $\tilde{f}_i : \mathcal{D}^{m+n} \to \mathcal{D}$, $i \in [1, r]$ are logical functions and $u_k(t) \in \mathcal{D}$, $k \in [1, m]$, are inputs. Besides, we assume the states of the other $n - r$ nodes are determined by the algebraic logical equations.

$$\begin{cases} x^1(t+1) = \tilde{L}u(t)x^1(t)x^2(t) \\ x^2(t) \in S(M_g)x^1(t), \quad x^1(t) \in \mathcal{X}^1 \end{cases} \quad (16)$$

$$x^1(t+1) \in \tilde{L}u(t)x^1(t)S(M_g)x^1(t) = \tilde{L}u(t)S(\bar{M}_g)x^1(t), \quad x^1(t) \in \mathcal{X}^1 \quad (17)$$

where $\tilde{L} = M_{\tilde{f}_1} * \ldots * M_{\tilde{f}_r}$, and $M_{\tilde{f}_i}$ is the structure matrix of logical function $\tilde{f}_i$, $i \in [1, r]$.

Definition 5: 1) Given a control sequence $U = \{u(0), u(1), \ldots, u(t-1)\}$ with $u(s) \in \Delta_{2^m}, s \in [0, t-1], t > 0$, and an initial state $x(0) = x_0 \in \mathcal{X}$, as well as $x(t) \in \mathcal{X}, t > 0$. If both $x(0)$ and $x(t)$ satisfy system (16), then $x(t)$ is called a solution to system (16) with respect to $x(0)$ and control sequence U.

2) DABCN (16) is called solvable [the solution to DABCN (16) is called unique] for the control sequence U if for any initial state $x(0) \in \mathcal{X}$, there exists a solution (unique solution) $x(t) \in \mathcal{X}, t > 0$, to system (16) with respect to $x(0)$ and control sequence U.

3) DABCN (16) is called solvable [the solution to DABCN (16) is called unique] if for any initial state $x(0) \in \mathcal{X}$, there exists a solution (unique solution) $x(t) \in \mathcal{X}, t > 0$, to system (16) with respect to $x(0)$ and arbitrary control sequence U.

Theorem 1: DABN (2) is solvable if and only if

$$LS(\bar{M}_g)x^1 \subset \mathcal{X}^1, \ \forall x^1 \in \mathcal{X}^1.$$

Theorem 2: The solution to DABN (2) is unique if and only if

$$LS(\bar{M}_g)x^1 \subset \mathcal{X}_0^1 \ \forall x^1 \in \mathcal{X}^1.$$

Since $\tilde{L} \in M_{2^r \times 2^{m+n}}$, we split $\tilde{L}$ into 2^m blocks as

$$\tilde{L} = [\mathrm{Blk}_1(L) \ \mathrm{Blk}_2(L) \ \cdots \ \mathrm{Blk}_{2^m}(L)] \tag{18}$$

where $\mathrm{Blk}_p(L) \in \mathcal{L}_{2^r \times 2^n}$, $p \in Q_m$. $u(t)$ can be regarded as a switching signal between $\mathrm{Blk}_p(L)$. From Definition 5 and Theorems 1 and 2, it is easy to derive the following result.

↓

Theorem 4: DABCN (16) is solvable if and only if for any $p \in Q_m$, one has

$$\mathrm{Blk}_p(L)S(\bar{M}_g)x^1 \subset \mathcal{X}^1 \ \ \forall x^1 \in \mathcal{X}^1. \tag{19}$$

Theorem 5: The solution to DABCN (16) is unique if and only if for any $p \in Q_m$, one has

$$\mathrm{Blk}_p(L)S(\bar{M}_g)x^1 \subset \mathcal{X}_0^1 \ \ \forall x^1 \in \mathcal{X}^1. \tag{20}$$

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