Time-delayed and Multi-valued Logical Systems

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日　　期:　August 21, 2020

Boolean Networks With Time Delays

Background:

- During the growth of a bacterium, several external environmental conditions including temperature, growth rate, external interference, or concentration of nutrition can cause time delays.
- Thus, BNs with time delays are sometimes better to model real biological systems or gene networks.
- Moreover, in many situations, time delay cannot be ignored in practice, since it can heavily affect the dynamics of the networks.

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I. Problem Formulation

A Boolean network of a set of nodes $A_1, A_2, \ldots A_n$ can be described as:

$$\begin{cases} A_1(t+1) = f_1(A_1(t),\ A_2(t), \ldots, A_n(t)), \\ A_2(t+1) = f_2(A_1(t),\ A_2(t), \ldots, A_n(t)), \\ \vdots \\ A_n(t+1) = f_n(A_1(t),\ A_2(t), \ldots, A_n(t)), \end{cases} \quad (1)$$

Since time delay cannot be avoided in many cases, we assume that **the Boolean networks have time-invariant** delays in states as follows:

$$\begin{cases} A_1(t+1) = f_1(A_1(t-\tau),\ A_2(t-\tau), \ldots, A_n(t-\tau)), \\ A_2(t+1) = f_2(A_1(t-\tau),\ A_2(t-\tau), \ldots, A_n(t-\tau)), \\ \vdots \\ A_n(t+1) = f_n(A_1(t-\tau),\ A_2(t-\tau), \ldots, A_n(t-\tau)), \end{cases} \quad (2)$$

⇒

$$\begin{aligned} A_i(t+1) &= M_i A_1(t-\tau) A_2(t-\tau) \cdots A_n(t-\tau) \\ &= M_i x(t-\tau), \quad i = 1, 2, \ldots, n. \end{aligned} \quad (3)$$

⇓

$$x(t+1) = L_1 x(t-\tau),$$

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I. Problem Formulation

Boolean control networks with time invariant integer delays in states as follows:

$$\begin{cases} A_1(t+1) = f_1(u_1(t), \ldots, u_m(t), A_1(t-\tau), \ldots, A_n(t-\tau)), \\ A_2(t+1) = f_2(u_1(t), \ldots, u_m(t), A_1(t-\tau), \ldots, A_n(t-\tau)), \\ \vdots \\ A_n(t+1) = f_n(u_1(t), \ldots, u_m(t), A_1(t-\tau), \ldots, A_n(t-\tau)), \end{cases} \quad (4)$$

Two kinds of controls are considered:

(1) The controls are logical variables satisfying certain logical rules, called input networks such as:

$$\begin{cases} u_1(t+1) = g_1(u_1(t),\ u_2(t), \ldots, u_m(t)), \\ u_2(t+1) = g_2(u_1(t),\ u_2(t), \ldots, u_m(t)), \\ \vdots \\ u_m(t+1) = g_n(u_1(t),\ u_2(t), \ldots, u_m(t)), \end{cases} \quad (5)$$

(2) The control is a free Boolean sequence. Precisely, set $u(t) = \ltimes_{j=1}^{m} u_j(t)$. Then the control is a designed sequence.

$$A_i(t+1) = M_{1i}u(t)x(t-\tau), \quad i = 1, 2, \ldots, n. \quad (6)$$

$$u_j(t+1) = M_{2j}u(t), \quad j = 1, 2, \ldots, m. \quad (7)$$

Multiply

$$\begin{cases} u(t+1) = Gu(t), \\ x(t+1) = Lu(t)x(t-\tau), \end{cases} \quad (8)$$

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II. Control via input Boolean networks

Case 1: s is fixed and *G* is fixed

Definition 1 Consider system (4) with control (5). Given initial state sequence $x(-\tau), x(-\tau+1), \ldots, x(0) \in \Delta_{2n}$, and the destination state x_d , x_d is said to be controllable from initial state $x(i-\tau)$, $(i \in \{0,1,\ldots,\tau\})$ at *s* steps with fixed (designable) input structure *G*, if we can find u_0 (and *G*) such that $x(s+i) = x_d$.

Theorem 1 Consider system (4) with control (5), where *G* is fixed. x_d is s step reachable from $x(i-\tau)$, $(i \in \{0,1,\ldots,\tau\})$, if and only if

$$x_d \in Col\{\Theta^G(s+i) W_{[2n,2m]} x(b-1-\tau)\}$$

where and hereafter "Col" is the column set, also there exist unique $a \in \{0, 1, 2, \ldots\}$ and $b \in \{1, 2, \ldots, \tau+1\}$ such that $s + i$ satisfies: $s + i = a(\tau + 1) + b$

$$\Theta^G(s+i)$$
$$= LG^{a(\tau+1)+(b-1)}\left(I_{2^m} \otimes LG^{(a-1)(\tau+1)+(b-1)}\right)$$
$$\times \left(I_{2^{2m}} \otimes LG^{(a-2)(\tau+1)+(b-1)}\right) \ldots \left(I_{2^{am}} \otimes LG^{(b-1)}\right) \times \left(I_{2^{(a-1)m}} \otimes \Phi_m\right) \ldots \left(I_{2^m} \otimes \Phi_m\right)\Phi_m,$$

where Φ_m is defined as $\Phi_m = \ltimes_{i=1}^{m} I_{2^{i-1}} \otimes [(I_2 \otimes W_{[2,2^{m-i}]}) M_r]$, $M_r = \delta_4[1, 4]$.

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II. Control via input Boolean networks

Case 1: s is fixed and *G* is fixed

Proof : A straightforward computation shows the following:

$$x(1) = Lu(0)x(-\tau),$$

$$\vdots$$

$$x(\tau+1) = Lu(\tau)x(0) = LG^{\tau}u(0)x(0),$$

$$x(\tau+2) = LG^{\tau+1}u(0)x(1) = LG^{\tau+1}(I_{2^m} \otimes L)\Phi_m u(0)x(-\tau),$$

$$\vdots$$

$$\begin{aligned} x(2(\tau+1)) &= LG^{2\tau+1}u(0)x(\tau+1) \\ &= LG^{2\tau+1}(I_{2^m} \otimes LG^{\tau})\Phi_m u(0)x(0), \end{aligned}$$

$$\vdots$$

$$\begin{aligned} 3(\tau+1)) &= LG^{3\tau+2}u(0)x(2\tau+2) \\ &= LG^{3\tau+2}u(0)LG^{2\tau+1}(I_{2^m} \otimes LG^{\tau})\Phi_m u(0)x(0) \\ &= LG^{3\tau+2}(I_{2^m} \otimes LG^{2\tau+1})(I_{2^{2m}} \otimes LG^{\tau}) \\ &\quad \times (I_{2^m} \otimes \Phi_m)\Phi_m u(0)x(0), \end{aligned}$$

Suppose that there exist unique $a \in \{0, 1, 2, \ldots\}$, $b \in \{1, 2, \ldots, \tau+1\}$ such that $s+i = a(\tau+1)+b$.

From the above analysis, using mathematical induction, we can prove that

$$\begin{aligned} x(s+i) &= x(a(\tau+1)+b) \\ &= LG^{a(\tau+1)+(b-1)}(I_{2^m} \otimes LG^{(a-1)(\tau+1)+(b-1)}) \\ &\quad \times (I_{2^{2m}} \otimes LG^{(a-2)(\tau+1)+(b-1)}) \cdots (I_{2^{am}} \otimes LG^{(b-1)}) \\ &\quad \times (I_{2^{(a-1)m}} \otimes \Phi_m) \cdots (I_{2^m} \otimes \Phi_m)\Phi_m u(0)x(b-1-\tau) \\ &= \Theta^G(s+i)u(0)x(b-1-\tau) \\ &= \Theta^G(s+i)W_{[2^n,2^m]}x(b-1-\tau)u(0). \end{aligned}$$

Notice the special form of $\Theta^G(s+i)W_{[2^n,2^m]}x(b-1-\tau)$ and $u(0)$, where $\Theta^G(s+i)W_{[2^n,2^m]}x(b-1-\tau)$ is a $2^n \times 2^m$ matrix, and its columns are elements in Δ_{2^n}, and $u(0) \in \Delta_{2^m}$, we can drive the conclusion.

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Notice that there are $m_0 = (2^m)^{2^m}$ possible distinct G, we can express each G in the condensed form and order them in "increasing order". Let us consider a subset $\Lambda \subset \{1,2, . . . , m_0\}$ and allow G be chosen from the admissible set $\{G_\lambda | \lambda \in \Lambda\}$.

Corollary 1 Consider system (4) with control (5), where $G \in \{G_\lambda | \lambda \in \Lambda\}$. Then x_d is s step reachable from $x(i-\tau)$, $(i \in \{0,1, . . . , \tau\})$, if and only if

$$x_d \in Col\{\Theta^{G_\lambda}(s+i)W_{[2n,2m]}x(b-1-\tau) | \lambda \in \Lambda\}$$

where there exist unique $a \in \{0, 1, 2, . . .\}$ and $b \in \{1, 2, . . . , \tau + 1\}$ such that $s + i$ satisfies:

$$s + i = a(\tau + 1) + b$$

$$\Theta^{G_\lambda}(s+i)$$

$$= LG^{a(\tau+1)+(b-1)}\left(I_{2^m} \otimes LG^{(a-1)(\tau+1)+(b-1)}\right)$$

$$\times \left(I_{2^{2m}} \otimes LG_\lambda{}^{(a-2)(\tau+1)+(b-1)}\right) \ldots \left(I_{2^{am}} \otimes LG_\lambda{}^{(b-1)}\right) \times \left(I_{2^{(a-1)m}} \otimes \Phi_m\right) \ldots (I_{2^m} \otimes \Phi_m)\Phi_m,$$

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III. Controllability via free Boolean sequence

Definition 2 Given initial state sequence $x(-\tau), x(-\tau+1), \dots, x(0) \in \Delta_{2n}$, and destination state x_d . The Boolean control network (4) is said to be controllable from $x(i-\tau)$, $(i \in \{0,1,\dots,\tau\})$ to x_d (by free Boolean sequence) at s steps, if we can find control $u(t)$, such that $x(s+i) = x_d$.

Define $\tilde{L} = LW_{[2n,2m]}$, notice that $x(t-\tau) \in R^{2^n}, u(t) \in R^{2^m}$ be two columns, then the second equation in (8) can be expressed as $x(t+1) = \tilde{L}x(t-\tau)u(t)$. It yields:

$$\begin{aligned} x(s+i) &= \tilde{L}x(s+i-1-\tau)u(s+i-1) \\ &= \tilde{L}^2x(s+i-2-2\tau)u(s+i-2-\tau)u(s+i-1) \\ &= \tilde{L}^3x(s+i-3-3\tau)u(s+i-3-2\tau) \\ &\quad \times u(s+i-2-\tau)u(s+i-1) \\ &= \cdots \\ &= \tilde{L}^kx(s+i-k-k\tau)u(s+i-k-(k-1)\tau) \\ &\quad \times u(s+i-(k-1)-(k-2)\tau)\cdots u(s+i-1). \end{aligned}$$

Assume that $s+i-k-k\tau = j-\tau$, where $j \in \{0, 1, \dots, \tau\}$.

Theorem 2 x_d is reachable from $x(i-\tau)$, $(i \in \{0,1,\dots,\tau\})$ at s steps by controls of Boolean sequences $u(s+i-k-(k-1)\tau)u(s+i-(k-1)-(k-2)\tau)\cdots u(s+i-1)$ if and only if

$$x_d \in \mathrm{Col}\{\tilde{L}^k x(j-\tau)\}$$

where there exists unique j and k such that $s+i-k-k\tau = j-\tau$, $j \in \{0, 1, \dots, \tau\}$.

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IV. Example

Consider the following Boolean control networks, its logical equation is

$$\begin{cases} A(t+1) = u(t) \wedge A(t-\tau), \\ B(t+1) = u(t) \vee B(t-\tau), \\ C(t+1) = u(t) \rightarrow C(t-\tau), \\ D(t+1) = D(t-\tau) \leftrightarrow E(t-\tau), \\ E(t+1) = E(t-\tau), \\ F(t+1) = F(t-\tau), \\ G(t+1) = G(t-\tau), \\ H(t+1) = H(t-\tau), \\ I(t+1) = I(t-\tau), \\ J(t+1) = \neg J(t-\tau). \end{cases}$$

with controls satisfying

$$u(t+1) = \neg u(t).$$

Denote $x(t) = A(t)B(t)C(t)D(t)E(t)F(t)G(t)H(t)I(t)J(t)$, then we can express system (10)–(11) as

$$\begin{cases} u(t+1) = Gu(t) \\ x(t+1) = Lu(t)x(t-\tau) \end{cases}$$

where $G = \delta_2[2, 1]$, $L = M_c(I_4 \otimes M_d)(I_{2^4} \otimes M_i)(I_2 \otimes W_{[2]})\Phi_1(I_2 \otimes W_{[2,4]})\Phi_1(I_{2^4} \otimes \otimes M_e)(I_{2^5} \otimes \Phi_1)(I_{2^{10}} \otimes M_n)$. L is a $2^{10} \times 2^{11}$ matrix, even using condensed form, it is still too long to show here, hence we do not write it specifically.

Assume that $s = 5$, $i = 2$, $\tau = 10$, $x_d = \delta_{1024}^2$, $x(-4) = \delta_{1024}^1$. Through calculation, it leads to

$$\Theta^G(s+i)W_{[2^n,2^m]}x(b-1-\tau)$$
$$= LG^6 W_{[2^{10},2]}x(-4) = \delta_{1024}[2, 514]$$

then, $x_d \in \mathrm{Col}\{\Theta^G(s+i)W_{[2^n,2^m]}x(b-1-\tau)\}$, using Theorem 1, we can see that x_d can be reached from $x(i - \tau)$ in five steps.

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End

Observability of BCN with State Time Delays

Problem Formulation

A Boolean network of a set of nodes $A_1, A_2, \ldots A_n$ can be described as:

$$\begin{cases} A_1(t+1) = f_1(u_1(t), \ldots, u_m(t), A_1(t-\tau), \ldots, A_n(t-\tau)) \\ A_2(t+1) = f_2(u_1(t), \ldots, u_m(t), A_1(t-\tau), \ldots, A_n(t-\tau)) \\ \vdots \\ A_n(t+1) = f_n(u_1(t), \ldots, u_m(t), A_1(t-\tau), \ldots, A_n(t-\tau)) \end{cases} \quad (1)$$

$$y_j(t) = h_j(A_1(t),\ A_2(t), \ldots, A_n(t)), \quad j = 1, 2, \ldots, p \quad (2)$$

Two kinds of controls are considered:

(1) The controls are logical variables satisfying certain logical rules, called input networks such as:

$$\begin{cases} u_1(t+1) = g_1(u_1(t),\ u_2(t), \ldots, u_m(t)), \\ u_2(t+1) = g_2(u_1(t),\ u_2(t), \ldots, u_m(t)), \\ \vdots \\ u_m(t+1) = g_n(u_1(t),\ u_2(t), \ldots, u_m(t)), \end{cases} \quad (3)$$

(2) The control is a free Boolean sequence. Precisely, set $u(t) = \ltimes_{j=1}^{m} u_j(t)$.Then the control is a designed sequence.

$\Rightarrow$

$$\begin{cases} u(t+1) = Gu(t) \\ x(t+1) = Lu(t)x(t-\tau) \end{cases}$$

$$y(t) = Hx(t)$$

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Observability of the Boolean Control Networks

Case 1: the controls are logical variables

Definition 1 The Boolean control network (1), (2) is observable if for the initial state sequence $x(-\tau), x(-\tau+1), \ldots, x(0) \in \Delta_{2^n}$, there exists finite time s, such that the initial state can be uniquely determined from the knowledge of the controls $\{u(0), u(1), \ldots, u(s)\}$ and the outputs $\{y(0), y(1), \ldots, y(s)\}$.

Define a sequence of matrices $\Gamma_j \in \mathcal{L}_{2^p \times 2^{n+m}}$, $j = a(\tau+1)+b$, where $a \in \{0, 1, 2\cdots\}$, $b \in \{1, 2, \ldots, \tau+1\}$, as

$$\Gamma_j = HLG^{a(\tau+1)+(b-1)} (I_{2^m} \otimes LG^{(a-1)(\tau+1)+(b-1)})(I_{2^{2m}} \otimes LG^{(a-2)(\tau+1)+(b-1)}) \cdots (I_{2^{am}} \otimes LG^{(b-1)})(I_{2^{(a-1)m}} \otimes \Phi_m) \cdots (I_{2^m} \otimes \Phi_m)\Phi_m$$

where H, L, G are the transition matrices of (2), (1), and (3) respectively. H, L, G, and Φ_m are as previously defined.

Split Γ_j into 2^m equal blocks as

$$\Gamma_j = [\Gamma_{j,1}, \Gamma_{j,2}, \ldots, \Gamma_{j,2^m}].$$

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Observability of the Boolean Control Networks

Case 1: the controls are logical variables

Theorem 1: Consider (1) and (2) with control (3), or equivalently (6), (7). Assume that $u(0) = \delta_{2^m}^i$, $i \in \{1, 2, \ldots, 2^m\}$. Equations (6), (7) are observable if and only if there exists finite time s, $s = c(\tau + 1)$, where c is a positive integer, such that

$$\mathrm{rank}(\mathcal{O}_0) = 2^n, \mathrm{rank}(\mathcal{O}_1) = 2^n, \ldots, \mathrm{rank}(\mathcal{O}_\tau) = 2^n$$

where

$$\mathcal{O}_0 = \begin{bmatrix} \Gamma_0 \\ \Gamma_{\tau+1,i} \\ \vdots \\ \Gamma_{(c-1)(\tau+1)+\tau+1,i} \end{bmatrix} \quad \mathcal{O}_1 = \begin{bmatrix} \Gamma_{1,i} \\ \Gamma_{\tau+2,i} \\ \vdots \\ \Gamma_{(c-1)(\tau+1)+1,i} \end{bmatrix}$$

$$\cdots \mathcal{O}_\tau = \begin{bmatrix} \Gamma_{\tau,i} \\ \Gamma_{2\tau+1,i} \\ \vdots \\ \Gamma_{(c-1)(\tau+1)+\tau,i} \end{bmatrix}$$

and $\Gamma_0 = H$, $i \in \{1, 2, \ldots, 2^m\}$.

Proof: Notice from the definition of Γ_j, and $u(0) = \delta_{2^m}^i$, a straightforward computation shows the following:

$$\begin{aligned}
y(0) &= Hx(0) = \Gamma_0 x(0) \\
&\vdots \\
y(\tau+1) &= HLG^\tau u(0)x(0) = \Gamma_{\tau+1}u(0)x(0) = \Gamma_{\tau+1,i}x(0) \\
&\vdots \\
y(2\tau+2) &= HLG^{2\tau+1}u(0)LG^\tau u(0)x(0) = \Gamma_{2\tau+2,i}x(0) \\
&\vdots \\
y(s-\tau) &= y((c-1)(\tau+1)+1) = \Gamma_{(c-1)(\tau+1)+1,i}x(-\tau) \\
&\vdots \\
y(s) &= y(c(\tau+1)) = \Gamma_{(c-1)(\tau+1)+\tau+1,i}x(0).
\end{aligned}$$

From the above analysis, we can see that

$$\mathcal{O}_1^T\mathcal{O}_1 x(-\tau) = \mathcal{O}_1^T \begin{bmatrix} y(1) \\ y(\tau+2) \\ \vdots \\ y((c-1)(\tau+1)+1) \end{bmatrix}.$$

It implies that $x(-\tau)$ can be determined uniquely by the outputs if and only if $\mathrm{rank}(\mathcal{O}_1) = 2^n$, i.e., $\mathcal{O}_1^T\mathcal{O}_1$ is nonsingular. In the same way, we can see that $x(1-\tau), \ldots, x(-1), x(0)$ can be determined uniquely by the outputs if and only if $\mathrm{rank}(\mathcal{O}_2) = 2^n, \ldots, \mathrm{rank}(\mathcal{O}_\tau) = 2^n$, $\mathrm{rank}(\mathcal{O}_0) = 2^n$.

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Example

Case 1: the controls are logical variables

Example 1: Consider the following Boolean control networks:

$$\begin{cases} A(t+1) = u_1(t) \wedge A(t-1) \\ B(t+1) = u_2(t) \vee B(t-1). \end{cases} \quad (8)$$

The outputs are

$$\begin{cases} y_1(t) = \neg A(t) \\ y_2(t) = \neg B(t) \end{cases} \quad (9)$$

with controls satisfying

$$\begin{cases} u_1(t+1) = \neg u_2(t) \\ u_2(t+1) = u_1(t). \end{cases} \quad (10)$$

Assume that $u(0) = \delta_4^2$.

Denote $x(t) = A(t)B(t)$, $u(t) = u_1(t)u_2(t)$, $y(t) = y_1(t)y_2(t)$, then we can convert the Boolean control networks (8)–(10) into

$$\begin{cases} x(t+1) = Lu(t)x(t-1) \\ u(t+1) = Gu(t) \end{cases}$$
$$y(t) = Hx(t)$$

where

$$L = \delta_4[1, 1, 3, 3, 1, 2, 3, 4, 3, 3, 3, 3, 3, 4, 3, 4],$$
$$G = \delta_4[3, 1, 4, 2], \quad H = \delta_4[4, 3, 2, 1].$$

By calculation, we have

$$y(0) = \delta_4[4, 3, 2, 1]x(0)$$
$$y(1) = \delta_4[4, 3, 2, 1]x(-1)$$
$$y(2) = \delta_4[4, 4, 2, 2]x(0).$$

In the same way, we have

$$y(3) = \Gamma_{3,2}x(-1) = \delta_4[2, 2, 2, 2]x(-1) \quad (11)$$
$$y(4) = \Gamma_{4,2}x(0) = \delta_4[2, 2, 2, 2]x(0) \quad (12)$$

and then we have

$$\mathcal{O}_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(\mathcal{O}_1) = 2^n = 4$. Also, rank $(\mathcal{O}_0) = 4$. Then, the system is observable.

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End

Observability of the Boolean Control Networks

Case 2: the controls are free Boolean sequence.

Define a sequence of matrices

$$\Gamma_j = HL(I_{2^m} \otimes L)(I_{2^{2m}} \otimes L)\cdots(I_{2^{(j-1)m}} \otimes L)$$

where $j \in \{1,2,\ldots\}$.

Split Γ_1 into 2^m equal blocks as

$$\Gamma_1 = [\Gamma_{11}, \Gamma_{12} \cdots, \Gamma_{12^m}].$$

Split Γ_2 into 2^{2m} equal blocks as

$$\Gamma_2 = [\Gamma_{211}, \cdots, \Gamma_{212^m}, \cdots, \Gamma_{22^m1}, \cdots, \Gamma_{22^m2^m}].$$

Doing it repeatedly, split Γ_j into 2^{jm} equal blocks as

$$\Gamma_j = [\Gamma_{j1\cdots1}, \cdots, \Gamma_{j1\cdots2^m}, \cdots, \Gamma_{j2^m\cdots2^m1}, \cdots, \Gamma_{j2^m\cdots2^m2^m}].$$

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Observability of the Boolean Control Networks

Case 2: the controls are free Boolean sequence.

Theorem 2: Consider (1) and (2). Assume that there is a free Boolean sequence of control as $u(0) = \delta_{2^m}^{i_0}$, $u(1) = \delta_{2^m}^{i_1}, \ldots$, where $i_0, i_1, \ldots \in \{1, 2, \ldots, 2^m\}$. Then the systems (1), (2) are observable if and only if there exists finite time s, $s = d(\tau + 1)$, and d is a positive integer such that

$$\operatorname{rank}(\mathcal{O}_0) = 2^n, \operatorname{rank}(\mathcal{O}_1) = 2^n, \ldots, \operatorname{rank}(\mathcal{O}_\tau) = 2^n$$

where

$$\mathcal{O}_0 = \begin{bmatrix} \Gamma_0 \\ \Gamma_{1i_\tau} \\ \Gamma_{2i_{2\tau+1}i_\tau} \\ \vdots \\ \Gamma_{di_{(d-1)(\tau+1)+\tau}\cdots i_\tau} \end{bmatrix} \mathcal{O}_1 = \begin{bmatrix} \Gamma_{1i_0} \\ \Gamma_{2i_{\tau+1}i_0} \\ \vdots \\ \Gamma_{di_{(d-1)(\tau+1)}\cdots i_0} \end{bmatrix}$$

$$\cdots \mathcal{O}_\tau = \begin{bmatrix} \Gamma_{1i_{\tau-1}} \\ \Gamma_{2i_{2\tau}i_{\tau-1}} \\ \vdots \\ \Gamma_{di_{(d-1)(\tau+1)+\tau-1}\cdots i_{\tau-1}} \end{bmatrix} \text{ and } \Gamma_0 = H.$$

Proof: In an algebraic form, a Boolean control networks (1), (2) can be expressed as

$$x(t+1) = Lu(t)x(t-\tau)$$
$$y(t) = Hx(t).$$

A straightforward computation shows the following:

$$y(0) = Hx(0) = \Gamma_0 x(0)$$
$$\vdots$$
$$y(\tau+1) = HLu(\tau)x(0) = \Gamma_1 u(\tau)x(0) = \Gamma_{1i_\tau}x(0)$$
$$\vdots$$
$$y((d-1)(\tau+1)+1) = \Gamma_{di_{(d-1)(\tau+1)}\cdots i_0}x(-\tau)$$
$$\vdots$$
$$y(s) = \Gamma_{di_{(d-1)(\tau+1)+\tau}\cdots i_\tau}x(0).$$

From the above analysis, we can see that

$$\mathcal{O}_0^T\mathcal{O}_0 x(0) = \mathcal{O}_0^T \begin{bmatrix} y(0) \\ y(\tau+1) \\ y(2(\tau+1)) \\ \vdots \\ y(d(\tau+1)) \end{bmatrix}.$$

It implies that $x(0)$ can be determined uniquely by the outputs if and only if $\operatorname{rank}(\mathcal{O}_0) = 2^n$, i.e., $\mathcal{O}_0^T\mathcal{O}_0$ is nonsingular. In the same way, we can see that $x(-\tau), x(1-\tau), \ldots, x(-1)$ can be determined uniquely by the outputs if and only if $\operatorname{rank}(\mathcal{O}_1) = 2^n, \ldots, \operatorname{rank}(\mathcal{O}_\tau) = 2^n$.

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Example

Case 2: the controls are free Boolean sequence.

Example 2: Consider the following Boolean control system:

$$\begin{cases} A(t+1) = u_1(t) \wedge B(t-1) \\ B(t+1) = A(t-1) \end{cases} \quad (13)$$

and the outputs are

$$y(t) = A(t) \to B(t). \quad (14)$$

Assume that $u(0) = \delta_2^1$, $u(1) = \delta_2^1$, $u(2) = \delta_2^1$ $u(3) = \delta_2^2$, $u(4) = \delta_2^2$, $u(5) = \delta_2^1$ $u(6) = \delta_2^1$, $u(7) = \delta_2^1$.

Denote $x(t) = A(t)B(t)$, $u(t) = u_1(t)$, then we can convert the Boolean networks (11), (12) into

$$\begin{cases} x(t+1) = Lu(t)x(t-1) \\ y(t) = Hx(t) \end{cases} \quad (15)$$

where

$$L = \delta_4[1, 3, 2, 4, 3, 3, 4, 4], \quad H = \delta_2[1, 2, 1, 1].$$

By calculation, it leads to

$$\mathcal{O}_0 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{O}_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

and

$$\text{rank}(\mathcal{O}_0) = 2^n = 4, \quad \text{rank}(\mathcal{O}_1) = 2^n = 4.$$

Then, systems (11), (12) are observable. And, we notice that the observability of the Boolean control network is related to the selection of the controls. For example, by letting $u \equiv \delta_2^2$, the Boolean control networks (11), (12) are not observable.

Li F , et al. Controllability of Boolean control networks with time delays in states [J], 2011

End

Some Necessary and Sufficient Conditions For The Output Controllability of Temporal Boolean Control Networks

I. Problem Formulation

$$a_i(t+1) = f_i(a_1(t), a_2(t), \ldots, a_n(t)), \;\; i = 1, 2, \ldots, n, \tag{3.1}$$

BN

$$a_i(t+1) = f_i(a_1(t), \ldots, a_n(t), a_1(t-1), \ldots, a_n(t-1), \ldots, a_1(t-\tau), \ldots, a_n(t-\tau)), \;\; i = 1, 2, \ldots, n, \tag{3.2}$$

Algebraic Form

$$\begin{aligned} a_i(t+1) &= M_i \ltimes_{j=1}^{n} a_j(t) \ltimes_{j=1}^{n} a_j(t-1) \ldots \ltimes_{j=1}^{n} a_j(t-\tau) \\ &= M_i x(t) x(t-1) \ldots x(t-\tau), \;\; i = 1, \ldots, n. \end{aligned} \tag{3.3}$$

Multiply

TBN

$$\begin{aligned} x(t+1) &= \ltimes_{i=1}^{n} a_i(t+1) \\ &= \ltimes_{i=1}^{n} [M_i x(t) x(t-1) \ldots x(t-\tau)]. \end{aligned} \tag{3.4}$$

$$L_0 = M_1[\ltimes_{i=2}^{n} I_{n(\tau+1)} \otimes M_i \Phi_{n(\tau+1)}]$$

$$x(t+1) = L_0 x(t) x(t-1) \ldots x(t-\tau) \tag{3.5}$$

Liu Y , et al. Some necessary and sufficient conditions for the output controllability of temporal Boolean control networks [J], 2014

Common methods of dimension extension

Recall a μth order Boolean network, its dynamics can be expressed as [25]:

$$\begin{cases} A_1(t+1) = f_1(A_1(t-\mu+1),\ldots,A_n(t-\mu+1),\ldots,A_1(t),\ldots,A_n(t)), \\ A_2(t+1) = f_2(A_1(t-\mu+1),\ldots,A_n(t-\mu+1),\ldots,A_1(t),\ldots,A_n(t)), \\ \vdots \\ A_n(t+1) = f_n(A_1(t-\mu+1),\ldots,A_n(t-\mu+1),\ldots,A_1(t),\ldots,A_n(t)), \\ t \geqslant \mu-1, \end{cases} \tag{1}$$

where $A_i \in \mathcal{D}$ and $f_i : \mathcal{D}^{\mu n} \to \mathcal{D}$, $i = 1,2,\ldots,n$ are logical functions; $t = 0,1,2,\ldots$ From (1), we can see that a μth order Boolean network is a very general Boolean networks with time delays.

In the following, we consider μth order Boolean control network as follows:

$$\begin{cases} A_1(t+1) = f_1(u_1(t-\mu+1),\ldots,u_m(t-\mu+1),A_1(t-\mu+1),\ldots,A_n(t-\mu+1),\ldots,A_1(t),\ldots,A_n(t)), \\ A_2(t+1) = f_2(u_1(t-\mu+1),\ldots,u_m(t-\mu+1),A_1(t-\mu+1),\ldots,A_n(t-\mu+1),\ldots,A_1(t),\ldots,A_n(t)), \\ \vdots \\ A_n(t+1) = f_n(u_1(t-\mu+1),\ldots,u_m(t-\mu+1),A_1(t-\mu+1),\ldots,A_n(t-\mu+1),\ldots,A_1(t),\ldots,A_n(t)), \quad t \geqslant \mu-1, \end{cases} \tag{2}$$

where $A_i, u_i \in \mathcal{D}$, u_i are controls (inputs) and f_i, $i = 1,2,\ldots,n$ are logical functions.

In order to convert (2) into algebraic form, we define $x(t) = \ltimes_{i=1}^{n} A_i(t) \in \Delta_{2^n}$, $u(t) = \ltimes_{i=1}^{m} u_i(t) \in \Delta_{2^m}$ and $z(t) = \ltimes_{i=t}^{t+\mu-1} x(i) \in \Delta_{2^{\mu n}}$. Assume that the structure matrix of f_i is $M_i \in \mathcal{L}_{2\times 2^{\mu n+m}}$. We can express (2) into

$$A_i(t+1) = M_i u(t-\mu+1) z(t-\mu+1), \quad i = 1,2,\ldots,n, \ t = \mu-1,\mu\ldots \tag{3}$$

[25] Cheng D, et al. Analysis and Control of Boolean Networks: A Semi-tensor Product Approach[M], 2011.

Common methods of dimension extension

Multiplying the equation in (3) together yields:

$$\begin{aligned} x(t+1) &= A_1(t+1)A_2(t+1)\cdots A_n(t+1) \\ &= M_1(I_{2^{m+\mu n}} \otimes M_2)\Phi_{m+\mu n}(I_{2^{m+\mu n}} \otimes M_3)\Phi_{m+\mu n}\cdots(I_{2^{m+\mu n}} \otimes M_n)\Phi_{m+\mu n}u(t-\mu+1)z(t-\mu+1). \end{aligned}$$

To see the definition of $\Phi_{m+\mu n}$ and its concerning results, we refer to [7]. Then, we can convert (2) into

$$x(t+1) = L_0 u(t-\mu+1)z(t-\mu+1), \tag{4}$$

where $L_0 = M_1(I_{2^{m+\mu n}} \otimes M_2)\Phi_{m+\mu n}(I_{2^{m+\mu n}} \otimes M_3)\Phi_{m+\mu n}\cdots(I_{2^{m+\mu n}} \otimes M_n)\Phi_{m+\mu n}$.

Using some properties of the semi-tensor product of matrices, we have

$$\begin{aligned} z(t+1) &= \ltimes_{i=t+1}^{t+\mu} x(i) = x(t+1)x(t+2)\cdots x(t+\mu-1)L_0 u(t)z(t) \\ &= x(t+1)x(t+2)\cdots x(t+\mu-1)L_0 u(t)x(t)\cdots x(t+\mu-1) = (I_{2^{(\mu-1)n}} \otimes L_0)W_{[2^{m+n},2^{(\mu-1)n}]}(I_{2^{m+n}} \otimes \Phi_{(\mu-1)n})u(t)z(t). \end{aligned}$$

This implies that

$$z(t+1) = Lu(t)z(t), \tag{5}$$

where $L = (I_{2^{(\mu-1)n}} \otimes L_0)W_{[2^{m+n},2^{(\mu-1)n}]}(I_{2^{m+n}} \otimes \Phi_{(\mu-1)n})$.

TBCN

Next, we consider TBCN with outputs as follows:

$$\begin{cases} a_i(t+1) = f_i(u_1(t), \dots u_m(t), a_1(t), \dots, a_n(t), a_1(t-1), \dots, a_n(t-1), \\ \qquad \dots, a_1(t-\tau), \dots, a_n(t-\tau)), \ i = 1, \dots, n, \\ y_j(t) = h_j(a_1(t), \dots, a_n(t)), \ j = 1, \dots, p, \end{cases} \tag{3.7}$$

where $u_i, i = 1, 2, \dots, m$ are inputs (or controls); $y_j(t), \ j = 1, \dots, p$ are outputs; $f_i, \ i = 1, \dots, n, \ h_j, \ j = 1, \dots, p$ are logical functions. In this paper, two kinds of inputs (or controls) are considered:

(A) The controls satisfy certain logical rules, called input networks such as:

$$u_i(t+1) = g_i(u_1(t), u_2(t), \dots, u_m(t)), \ i = 1, 2, \dots, m, \tag{3.8}$$

where $g_i, i = 1, 2, \dots, m$ are logical functions, and the initial states $u_j(0), j = 1, 2, \dots, m$, can be arbitrarily given.

(B) The controls are free Boolean sequences (or designable).

$$\begin{cases} x(t+1) = Lu(t)x(t)x(t-1)\dots x(t-\tau), \\ y(t) = Hx(t), \end{cases} \tag{3.12}$$

$$\text{and } u(t+1) = Gu(t), \tag{3.13}$$

where L, H are respectively the network transition matrices of two equations in (3.7), and G is the network transition matrix of (3.8).

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II. Output controllability of input Boolean networks

Definition 4.1. Consider the TBCN (3.12) with control (3.13). Given the finite time $s \in \mathbb{N}^+$, initial state sequence $x(-i)$, $i \in \{0, 1, \dots, \tau\}$ and the destination output $y_f \in \Delta_{2^p}$, y_f is said to be $s-output-controllable$ (*or reachable*) from initial state sequence with fixed (designable) input structure G, if we can find control input $u(0)$ (and G), such that $y(s) = y_f$.

Definition 4.2. The TBCN (3.12) with control (3.13) is said to be $s-output-controllable$ (*or reachable*) if for any $a_i \in \Delta_{2^n}$, $i \in \{0, 1, \dots, \tau\}$ and $b \in \Delta_{2^p}$, there exist the finite time $s \in \mathbb{N}^+$ and the control input $u(0)$ such that $x(-i) = a_i$, $i \in \{0, 1, \dots, \tau\}$ to $y(s) = b$.

Theorem 4.6. *Consider the TBCN (3.12) with control (3.13), where G is fixed. y_f is s-output-reachable from $x(-i)$, $i \in \{0, 1, \dots, \tau\}$, if and only if $y_f^\top H L_s^G W_{[2^{n(\tau+1)}, 2^m]} X(\tau) \neq \underbrace{(0, \dots, 0)}_{2^m}^\top$ or equivalently, there exists at least one entry of $y_f^\top H L_s^G W_{[2^{n(\tau+1)}, 2^m]} X(\tau)$ equaling 1, where*

$$L_t^G = \begin{cases} L, & t = 1, \\ LG[(I_{2^m} \otimes L_1^G)\Phi_m][I_{2^m} \otimes W_{2^{n\tau}, 2^{n(\tau+1)}} \Phi_{n(\tau)}], & t = 2, \\ LG^{s-1}[(I_{2^m} \otimes L_{s-1}^G)\Phi_m][\ltimes_{i=s-2}^{1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})] \\ \qquad \ltimes [I_{2^m} \otimes W_{[2^{n(\tau-s+2)}, 2^{n(\tau+1)}]} \Phi_{n(\tau-t+2)}], & s = 3, \dots, \tau+1, \\ LG^{s-1}[(I_{2^m} \otimes L_{s-1}^G)\Phi_m][\ltimes_{i=s-2}^{s-\tau-1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})], & s > \tau+1. \end{cases} \quad (4.4)$$

If it is the ith entry, then $u(0) = \delta_{2^m}^i$.

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Corollary 4.8. *Consider the TBCN (3.12) with control (3.13), where* $G \in \{G_\lambda | \lambda \in \Lambda\}$*. Then* y_f *is s-output-reachable from* $x(-i)$, $i \in \{0, 1, \ldots, \tau\}$*, if and only if there exists at least one entry of* $\{y_f^\top H L_s^{G_\lambda} W_{[2^{n(\tau+1)}, 2^m]} X(\tau) | \lambda \in \Lambda\}$ *equaling 1, where* L_s^G *is given by (4.4).*

Proposition 4.9. *The number of different controls* $u(0)$ *that steer TBCNs (3.12) with control (3.13) from* $x(-i)$, $i \in \{0, 1, \ldots, \tau\}$ *to* $y(s) = y_f$ *in s time steps is*

$$l(s; X(\tau), y_f) = y_f^\top Q_s X(\tau), \quad s \in \mathbb{N}^+,$$

where $Q_s = H L_s^G 1_{2^m}$, $1_{2^m} = \underbrace{[1, \ldots, 1]}_{2^m}^\top$ *and* L_s^G *is given by (4.4).*

Theorem 4.10. *The TBCN (3.12) with control (3.13) is s-output-controllable if and only if all the entries of* Q_s *are different from zero.*

Algorithm 4.11. *Assume the TBCN is given with its logical expression as (3.7) and input networks as (3.8).*

(A) *Convert (3.7) and (3.8) into a linear discrete time delay system as (3.12) and (3.13) such that G, L, H can be expressed by matrices.*

(B) *Compute* L_s^G *by (4.4).*

(C) *Get* $l(s; X(\tau), y(s)) = y(s)^\top Q_s X(\tau)$ *to see the number of different controls* $u(0)$ *that steers the TBCN from* $X(\tau)$ *to* $y(s)$*. If* $l(s; X(\tau), y(s)) = 0$*, it means there is no existence of such* $u(0)$*, then stop.*

(D) *Find which entry of vector* $y(s)^\top H L_s^G W_{[2^{n(\tau+1)}, 2^m]} X(\tau)$ *equals 1. If it is the 1st one, then* $u(0) = \delta_{2^m}^1$*. Similarly, if the ith one, then* $u(0) = \delta_{2^m}^i$*.*

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III. Control via free Boolean sequence

Definition 4.13. Given initial state $x(-i)$, $i \in \{0, 1, \ldots, \tau\}$, the destination output $y_f \in \Delta_{2^p}$ and the finite time $s \in \mathbb{N}^+$, the TBCN (3.12) is said to be $s-output-controllable$ (*or reachable*) from initial state $x(-i)$, $(i \in 0, 1, \ldots, \tau)$ to y_f (by free Boolean sequence), if we can find the control inputs $\{u(0), u(1), \ldots, u(s-1)\}$ such that $y(s) = y_f$.

Definition 4.14. The TBCN (3.12) is said to be $s-output-controllable$ (*or reachable*) if for any $a_i \in \Delta_{2^n}$, $i \in \{0, 1, \ldots, \tau\}$ and $b \in \Delta_{2^p}$, there exist the finite time $s \in \mathbb{N}^+$ and the control input $u(t)$ steers the TBCN from $x(-i) = a_i$, $i \in \{0, 1, \ldots, \tau\}$ to $y(s) = b$.

Theorem 4.15. *Consider TBCN (3.12). y_f is s-output-reachable from $x(-i)$, $i \in \{0, 1, \ldots, \tau\}$ by controls of Boolean sequences $U(s-1)$ of length s if and only if there exists at least one entry of $y_f^\top H \tilde{L}_s X(\tau)$ equaling 1, where*

$$\tilde{L}_s = \begin{cases} \tilde{L}, & s = 1 \\ \tilde{L}\tilde{L}_1 W_{[2^{n\tau}, 2^{m+n(\tau+1)}]} \Phi_{n\tau}, & s = 2, \\ \tilde{L}\tilde{L}_{s-1}[\ltimes_{i=s-2}^{1}(W_{[2^n, 2^{(s-1)m+n(\tau+1)}]} \tilde{L}_i \Phi_{im+n(\tau+1)})] & \\ \qquad \ltimes W_{[2^{(\tau+2-s)n}, 2^{(s-1)m+n(\tau+1)}]} \Phi_{(\tau+2-s)n}, & s = 3, \ldots, \tau+1. \\ \tilde{L}\tilde{L}_{s-1}[\ltimes_{i=s-2}^{s-\tau-1}(W_{[2^n, 2^{(s-1)m+n(\tau+1)}]} \tilde{L}_i \Phi_{im+n(\tau+1)})], & s > \tau+1. \end{cases} \tag{4.16}$$

If it is the ith one, then $U(s-1) = \delta_{2^{sm}}^i$.

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Proposition 4.17. *The number of different controls $u(t)$ that steer TBCN (3.12) from $x(-i)$, $i \in \{0, 1, \ldots, \tau\}$ to $y(s) = y_f$ in s time steps is*

$$l'(s; X(\tau), y_f) = y_f^\top P_s X(\tau), \ \ s \in \mathbb{N}^+,$$

where $P_s = H\tilde{L}_s W_{[2^{sm}, 2^{n(\tau+1)}]} 1_{2^{sm}}$ and $\tilde{L}_s$ is given by (4.16).

Theorem 4.18. *The TBCN (3.12) is s − output − controllable (or reachable) if and only if all the entries of P_s are different from zero.*

Algorithm 4.20. *If the TBCN is given with its logical expression as (3.7).*

(A) *Convert (3.7) into a linear discrete time delay system as (3.12) such that L, H can be expressed by matrices.*
(B) *Compute $\tilde{L}_s$ by (4.16).*
(C) *Get $l'(s; X(\tau), y(s)) = y_f^\top P_s X(\tau)$ to see the number of different controls $\{u(0), u(1), \ldots, u(s-1)\}$ that steer the TBCN from $X(\tau)$ to $y(s)$. If $l(s; X(\tau), y(s)) = 0$, it means there is no such control, then stop.*
(D) *Find which entry of vector $y(s)^\top H\tilde{L}_s X(\tau)$ equals 1. If it is the ith one, then $U(s-1) = \delta_{2^{sm}}^i$.*

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Controllability and Observability of BCN With Time-Variant Delays in States

- The constructed forest
- Matrix testing criteria
- Control design algorithms

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I. The definition of controllability

Definition 1: Consider (5). For any given initial time t_0, any given time delay function, any given initial state sequence $X_0 = (x(t_0 - \tau(t_0)), x(t_0 - \tau(t_0) + 1), \ldots, x(t_0)) \in (\Delta_{2^n})^{(\tau(t_0)+1)}$, any given destination state $x_d \in \Delta_{2^n}$, and any given $s \in \mathcal{N} \setminus \{0\}$.

1) x_d is said to be reachable from X_0 at the sth step if a control sequence $\{u(t_0), u(t_0+1), \ldots, u(t_0+s-1)\} \subset \Delta_{2^m}$ can be found such that the trajectory of (5) satisfies $x(t_0+s) = x_d$.
2) The set of all states that are reachable from X_0 at the sth step is said to be the s-step reachable set of X_0, denoted by $R_s(X_0)$.
3) The set of all states that are reachable from X_0 is said to be the reachable set of X_0, denoted by $R(X_0)$. Clearly $R(X_0) = \cup_{i \in \mathcal{N} \setminus \{0\}} R_i(X_0)$.
4) System (5) is said to be controllable from X_0 if $R(X_0) = \Delta_{2^n}$.
5) System (5) is said to be (globally) controllable if it is controllable from any $X_0 \in (\Delta_{2^n})^{(\tau(t_0)+1)}$.

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II. Controllability

Definition 2: A directed graph $G(V, E)$ is said to be the constructed forest of (5) if the vertex set $V = \{t \in \mathcal{Z} : t \geq t_0 - \tau(t_0)\}$, i.e., the time sequence of (5), and the edge set $E = \{(t', t'') : t' = t'' - 1 - \tau(t'' - 1)\} \subset V \times V$.

We identify each vertex t with the state $x(t)$ of (5) hereinafter. The following lemma holds.

Lemma 1: The constructed forest G consists of $\tau(t_0) + 1$ directed trees whose root set is $\{t_0 - \tau(t_0), t_0 - \tau(t_0) + 1, \ldots, t_0\}$.

The constructed forest can be obtained easily according to the following procedure: 1) draw the roots $t_0 - \tau(t_0), t_0 - \tau(t_0)+1, \ldots, t_0$; 2) draw the vertex t_0+1 and the edge pointing to it; 3) draw the vertex $t_0 + 2$ and the edge pointing to it; 4) see Fig. 1.

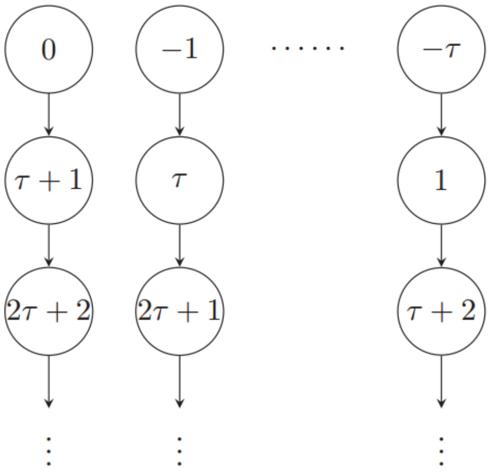

Fig. 1. Constructed forest of (5) with $\tau(t)$ constant that is studied in [7], where the number in each circle denotes the time step.

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II. Controllability

For the sake of discussion, some notations are presented.

1) Let $\{T_{t_0-\tau(t_0)}, T_{t_0-\tau(t_0)+1}, \ldots, T_{t_0}\}$ be the constructed forest $G(V, E)$.
2) Let $P_{t_0-\tau(t_0)+i}$, and N_i be any one given longest path of the tree $T_{t_0-\tau(t_0)+i}$ and the length of $P_{t_0-\tau(t_0)+i}$, respectively, $i = 0, 1, \ldots, \tau(t_0)$ (If $P_{t_0-\tau(t_0)+j}$ has an infinite number of vertices, set $N_j = +\infty$, $j = 0, 1, \ldots, \tau(t_0)$.).
3) Let P_c and N_c be any one given longest path in $\{P_{t_0-\tau(t_0)}, P_{t_0-\tau(t_0)+1}, \ldots, P_{t_0}\}$ and the length of P_c, respectively.

We call P_c a controllability constructed path since it can be used to characterize the global controllability of the system, as will be shown later. For example, in Fig. 1, the time sequence $\{0, \tau + 1, 2\tau + 2, \ldots\}$ is a controllability constructed path.

Remark 1: A controllability constructed path has either finite vertices or countably infinite vertices. In particular, if $\tau(t)$ is bounded, any controllability constructed path generated by $\tau(t)$ has countably infinite vertices. In fact, if it is not true, there must exist one vertex that has countably infinite sons, which infers that $\tau(t)$ is unbounded. The converse is not true. A counterexample is the following time delay function:

$$\tau^*(t) = \begin{cases} 1, & \text{if } t \geq t_0 = 0 \text{ is even} \\ t, & \text{if } t \geq t_0 = 0 \text{ is odd} \end{cases} \tag{6}$$

where $\tau^*(t)$ is obviously unbounded, but the unique controllability constructed path $\{-1, 1, 3, 5, \ldots\}$ has countably infinite vertices (Fig. 2).

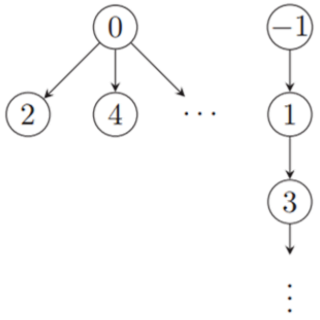

Fig. 2. Constructed forest generated by the time delay function (6), where the number in each circle denotes the time step.

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II. Controllability

Denote by

$$\{\mathbf{t}_0, t_1, \ldots, t_N\}(\text{or } \{\mathbf{t}_0, t_1, \ldots\}) \subset \mathcal{Z} \tag{7}$$

one of the controllability constructed paths of (5), where $t_0 - \tau(t_0) \le \mathbf{t}_0 \le t_0$, $t_{i+1} > t_i$ for all $i \ge 0$. Then, we identify the controllability constructed path (7) with the following subsystem (8) of (5). In (8), we still use t_0 to denote $\mathbf{t}_0$ when confusion does not occur

$$\begin{cases} x(t_{k+1}) = Lu(t_{k+1} - 1)x(t_k) \\ y(t_k) = Hx(t_k) \end{cases} \tag{8}$$

where $k \in \{0, 1, \ldots, N-1\}$ if (7) has a finite length N, and $k \in \mathcal{N}$ if (7) has length $+\infty$, $x \in \Delta_{2^n}$, $u \in \Delta_{2^m}$, L, H are the same as those in (5). In fact, it is easy to see that any path of (5) with its root in the set $\{t_0 - \tau(t_0), t_0 - \tau(t_0) + 1, \ldots, t_0\}$ has the same form as (8).

Note that (8) is a system with no time delays if the subscript of t is regarded as its time sequence. Then, similar to Fig. 1 of [8], an input-state dynamic graph of (8) can be illustrated intuitively, which implies that the reachable set of the state $x(t_0)$ satisfies $R(x(t_0)) = \cup_{s=1}^{\min\{2^{n+m}-1, N\}} R_s(x(t_0))$. Similarly, based on the concept of the input-state incidence matrix introduced in [8] and [8, Th. 3.3], the controllability criteria for (8) are easy to obtain as follows: In addition, Theorem 4 is a corollary of Theorem 1 of [12]. Following the idea of the proof of Theorem 1 of [12] ([12, Eq. (9)]), we have $R_s(x(t_0)) = \{Lu(t_s - 1)Lu(t_{s-1} - 1)\ldots Lu(t_1 - 1)x(t_0) : u(t_1 - 1), u(t_2 - 1), \ldots, u(t_s - 1) = \delta_{2^m}^1, \delta_{2^m}^2, \ldots, \delta_{2^m}^{2^m}\}$.

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II. Controllability

Theorem 4: Consider (8). Let $L\mathbf{1}_{2^m} := M$. Then:

1) $\delta_{2^n}^i$ is reachable from $\delta_{2^n}^j$ at the sth step if and only if $(M^s)_{ij} > 0$;
2) (8) is controllable from $\delta_{2^n}^j$ if and only if all the entries of $\mathrm{Col}_j(\sum_{k=1}^{\min\{2^{n+m}-1,N\}} M^k)$ are positive;
3) (8) is controllable if and only if all the entries of $\sum_{k=1}^{\min\{2^{n+m}-1,N\}} M^k$ are positive,

where N is the length of (7).

Based on the above preliminaries, our first main result on testing the controllability of (5) is obtained.

Theorem 5: Consider (5). Let $L\mathbf{1}_{2^m} := M$, and set

$$M_{t_0-\tau(t_0)+s} = \begin{cases} \sum_{k=1}^{\min\{N_s,2^{n+m}-1\}} M^k, & \text{if } N_s > 0 \\ \mathbf{0}_{2^n\times 2^n}, & \text{if } N_s = 0 \end{cases}$$

$s = 0, 1, \ldots, \tau(t_0)$. Then:

1) $\delta_{2^n}^i$ is reachable from $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \ldots, \delta_{2^n}^{i_{\tau(t_0)}})$ if and only if $\sum_{l=0}^{\tau(t_0)} (M_{t_0-\tau(t_0)+l})_{i,i_l} > 0$;
2) (5) is controllable from $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \ldots, \delta_{2^n}^{i_{\tau(t_0)}})$ if and only if $\sum_{l=0}^{\tau(t_0)} (M_{t_0-\tau(t_0)+l})_{i,i_l} > 0$ for all $i = 1, 2, \ldots, 2^n$;
3) (5) is controllable if and only if (8) is controllable;
4) (5) is controllable if and only if all the entries of $\sum_{k=1}^{\min\{N_c,2^{n+m}-1\}} M^k$ are positive.

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II. Controllability

Theorem 5: Consider (5). Let $L\mathbf{1}_{2^m} := M$, and set

$$M_{t_0-\tau(t_0)+s} = \begin{cases} \sum_{k=1}^{\min\{N_s,2^{n+m}-1\}} M^k, & \text{if } N_s > 0 \\ \mathbf{0}_{2^n\times 2^n}, & \text{if } N_s = 0 \end{cases}$$

$s = 0, 1, \ldots, \tau(t_0)$. Then:

1) $\delta_{2^n}^i$ is reachable from $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \ldots, \delta_{2^n}^{i_{\tau(t_0)}})$ if and only if $\sum_{l=0}^{\tau(t_0)} (M_{t_0-\tau(t_0)+l})_{i,i_l} > 0$;
2) (5) is controllable from $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \ldots, \delta_{2^n}^{i_{\tau(t_0)}})$ if and only if $\sum_{l=0}^{\tau(t_0)} (M_{t_0-\tau(t_0)+l})_{i,i_l} > 0$ for all $i = 1, 2, \ldots, 2^n$;
3) (5) is controllable if and only if (8) is controllable;
4) (5) is controllable if and only if all the entries of $\sum_{k=1}^{\min\{N_c,2^{n+m}-1\}} M^k$ are positive.

Proof :

Conclusion (1): By the constructed forest of (5) and Lemma 1, $\delta_{2^n}^i$ is reachable from $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \ldots, \delta_{2^n}^{i_{\tau(t_0)}})$ if and only if there exists a $0 \le k \le \tau(t_0)$ such that $\delta_{2^n}^i$ is reachable from $\delta_{2^n}^{i_k}$ in subsystem $P_{t_0-\tau(t_0)+k}$. Hence the conclusion holds by Theorem 4.

Conclusion (2): It is a direct corollary of Conclusion (1).

Conclusion (3): The "IF" part holds by Definiton 1. For the "ONLY IF" part, if (8) (a controllability constructed path) is not controllable, then nor is any one of the paths of the related constructed forest. Hence (5) is not controllable.

Conclusion (4): For the "IF" part, if all the entries of $\sum_{k=1}^{\min\{N_c,2^{n+m}-1\}} M^k$ are positive, then any one of the controllability constructed paths is controllable. Hence (5) is controllable. For the "ONLY IF" part, if $(\sum_{k=1}^{\min\{N_c,2^{n+m}-1\}} M^k)_{i',i''} = 0$ for some $0 \le i', i'' \le 2^n$, then $\delta_{2^n}^{i'}$ is not reachable from $\underbrace{(\delta_{2^n}^{i''}, \ldots, \delta_{2^n}^{i''})}_{\tau(t_0)+1}$.

Thus (5) is not controllable. ■

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II. Controllability

Corollary 1

Consider (5) with $\tau(t)$ constant. Let $L\mathbf{1}_{2^m} := M$. Then:

1) $\delta_{2^n}^{i}$ is reachable from $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \ldots, \delta_{2^n}^{i_{\tau(t_0)}})$ if and only if $(\sum_{k=1}^{2^{n+m}-1} M^k)_{i,i_l} > 0$ for some $0 \leq l \leq \tau(t_0)$;
2) (5) is controllable from $X_0 = (\delta_{2^n}^{i_0}, \delta_{2^n}^{i_1}, \ldots, \delta_{2^n}^{i_{\tau(t_0)}})$ if and only if $\sum_{l=0}^{\tau(t_0)} (\sum_{k=1}^{2^{n+m}-1} M^k)_{i,i_l} > 0$ for all $i = 1, 2, \ldots, 2^n$;
3) (5) is controllable if and only if all the entries of $\sum_{k=1}^{2^{n+m}-1} M^k$ are positive.

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III. Example

Consider the following Boolean control network:

$$\begin{cases} A(t+1) = u_1(t) \to A(t-\tau) \\ B(t+1) = u_2(t) \vee B(t-\tau). \end{cases} \tag{9}$$

Denote $x(t) = A(t)B(t)$, and $u(t) = u_1(t)u_2(t)$; then its algebraic form is $x(t+1) = Lu(t)x(t-\tau)$, where $L = M_i(I_4 \otimes M_d)(I_2 \otimes W_{[2]}) = \delta_4[1, 1, 3, 3, 1, 2, 3, 4, 1, 1, 1, 1, 1, 2, 1, 2]$, where $M_i = \delta_2[1, 2, 1, 1]$, $M_d = \delta_2[1, 1, 1, 2]$.

The time delay function is time-invariant, so it is bounded, then the controllability constructed path of (9) has countably infinite vertices. After replacing all the nonzero entries of $\sum_{k=1}^{2^4-1} M^k$ by 1, we get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

By (3) of Corollary 1, (9) is not controllable for any nonnegative constant time delay τ. Let τ be 2, the initial state sequence be $(\delta_4^1, \delta_4^2, \delta_4^3)$, and the destination state be δ_4^2 respectively. By (1) of Corollary 1, set $i = 2$, $i_0 = 1$, $i_1 = 2$, $i_2 = 3$. Since the $(2, 2)$th entry of matrix (10) is positive, δ_4^2 is reachable from $(\delta_4^1, \delta_4^2, \delta_4^3)$. By (2) of Corollary 1, (9) is controllable from $(\delta_4^4, \delta_4^4, \delta_4^4)$.

Note that from Example 1, we see that even if the constant time delay τ is unknown, we can use Corollary 1 to verify the global controllability of (5) with a constant time delay. While Theorem 3.5 of [7] can only verify the reachability, it cannot verify the global controllability if τ is unknown.

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End

IV. Control Design Algorithm

Algorithm 6: Since $\delta_{2^n}^i \in R(\delta_{2^n}^j)$, by Theorem 4, $(M^p)_{ij} > 0$ for some p.

1) *Step 1:* Find one (or the smallest) s, such that $(M^s)_{ij} > 0$, set $x(t_0) = \delta_{2^n}^j$, $x(t_s) = \delta_{2^n}^i$, go to Step 2.
2) *Step 2:* If $s = 1$, find one l, such that $(L\delta_{2^m}^l)_{ij} > 0$, set $u(t_s - 1) = \delta_{2^m}^l$, stop. Else, find one k, such that $M_{ik} > 0$, $(M^{s-1})_{kj} > 0$, set $x(t_{s-1}) = \delta_{2^n}^k$; find one l, such that $(L\delta_{2^m}^l)_{ik} > 0$, set $u(t_s - 1) = \delta_{2^m}^l$; set $s = s-1$, $i = k$, go back to Step 2.

Theorem 7: For (8), assume that x_d and x_0 are two states, and $x_d \in R_s(x_0)$. Then the control sequence $\{u(t_1 - 1), u(t_2 - 1), \ldots, u(t_s - 1)\}$ generated by Algorithm 6 drives x_0 to x_d, and the corresponding trajectory is the state sequence $\{x(t_0) = x_0, x(t_1), \ldots, x(t_s) = x_d\}$ generated by Algorithm 6.

Theorem 8: For (5), assume that $x_d \in R(X_0)$, where x_d and X_0 are the destination state and the initial state sequence, respectively. So, there exists a path in the form of (8), where the initial time step is $\mathbf{t}_0$ such that $t_0 - \tau(t_0) \le \mathbf{t}_0 \le t_0$, $x(\mathbf{t}_0) \in X_0$, and $x_d \in R_s(x(\mathbf{t}_0))$ for some positive integer s. Then the control sequence of (5) driving X_0 to x_d and the corresponding controlled trajectory can be constructed as follows.

1) Use Algorithm 6 to generate the control sequence $\{u(t_1 - 1), u(t_2 - 1), \ldots, u(t_s - 1)\} := U_{\text{Algorithm}}$ driving $x(\mathbf{t}_0)$ to x_d.
2) Set $u(t) = \delta_{2^m}^1$, where $t \in \{l \in \mathcal{Z} : t_0 \le l \le t_s - 1\} \setminus \{t_k - 1 : k = 1, 2, \ldots, s\}$; and use the symbol U_{free} to denote the set of these controls.
3) Use the symbol U to denote the control sequence $U_{\text{Algorithm}} \cup U_{\text{free}}$ that drives X_0 to x_d, and use U and X_0 to calculate the controlled trajectory $X_0, \ldots, x_d$.

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V. Observability

Definition 3: Consider (4). For any given initial time t_0 and any given time delay function, (4) is said to be observable if for each initial state sequence $(x(t_0 - \tau(t_0)), x(t_0 - \tau(t_0) + 1), \ldots, x(t_0)) \in \mathcal{D}^{n(\tau(t_0)+1)}$, there exists a control sequence $\{u(t_0), u(t_0+1), \ldots\} \subset \mathcal{D}^m$ such that the initial state sequence can be uniquely determined by the output sequence $\{y(t_0), y(t_0+1), \ldots\} \subset \mathcal{D}^q$.

According to Definition 3, it is easy to see that a controllability constructed path of (4) cannot generally determine its observability, while the set $\{P_{t_0-\tau(t_0)}, P_{t_0-\tau(t_0)+1}, \ldots, P_{t_0}\}$ can. Hence we call the set $\{P_{t_0-\tau(t_0)}, P_{t_0-\tau(t_0)+1}, \ldots, P_{t_0}\}$ a set of observability constructed paths. Then by Definition 3 and (8), the following theorem holds:

Theorem 9: System (4) is observable if and only if each subsystem of the set $\{P_{t_0-\tau(t_0)}, P_{t_0-\tau(t_0)+1}, \ldots, P_{t_0}\}$ is observable.

Now let P_o and N_o be any one given shortest path of the set $\{P_{t_0-\tau(t_0)}, P_{t_0-\tau(t_0)+1}, \ldots, P_{t_0}\}$ and the length of P_o, respectively. P_o is called an observability constructed path, since it determines the observability of (5) (See Theorem 10).

Theorem 10: System (4) is observable if and only if system P_o is observable.

Proof: Based on the above discussion, if system P_o is observable, then system $P_{t_0-\tau(t_0)+i}$ is observable, $i = 0, 1, \ldots, \tau(t_0)$. Hence by Theorem 9, (4) is observable. If system P_o is not observable, then (4) is not observable by Theorem 9. ■

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V. Observability

Definition 4: System (4) is said to be strongly controllable if system P_o is controllable.

Define a sequence of sets of matrices $\Gamma_i \subset \mathcal{L}_{2^q \times 2^n}$.

$$\begin{aligned}
\Gamma_0 &= \{H\} \\
\Gamma_1 &= \{HL\delta_{2^m}^i : i = 1, 2, \ldots, 2^m\} \\
&\vdots \\
\Gamma_s &= \left\{HL\delta_{2^m}^{i_1} L\delta_{2^m}^{i_2} \ldots L\delta_{2^m}^{i_s} : i_1, i_2, \ldots, i_s = 1, 2, \ldots, 2^m\right\} \\
&\vdots
\end{aligned} \tag{16}$$

In [10], it was proved that there exists s^* such that

$$\Gamma_s \subset \cup_{k=0}^{s^*} \Gamma_k, \quad \forall s > s^*. \tag{17}$$

Denote by O_i a matrix consisting of the elements in Γ_i and arranged in a column. Precisely

$$O_0 = H,\ O_1 = \begin{bmatrix} HL\delta_{2^m}^1 \\ HL\delta_{2^m}^2 \\ \vdots \\ HL\delta_{2^m}^{2^m} \end{bmatrix}, \ldots, O_{s^*} = \begin{bmatrix} HL\delta_{2^m}^1 \ldots L\delta_{2^m}^1 \\ HL\delta_{2^m}^1 \ldots L\delta_{2^m}^2 \\ \vdots \\ HL\delta_{2^m}^{2^m} \ldots L\delta_{2^m}^{2^m} \end{bmatrix}. \tag{18}$$

Theorem 11: Assume system P_o is controllable and $N_o \geq \frac{2^{n+m}(2^{m(s^*+1)}-1)}{2^m-1}$. Then it is observable if and only if

$$\text{rank}\left[O_0^T, O_1^T, \ldots, O_{s^*}^T\right]^T = 2^n. \tag{19}$$

Theorem 12: Assume that (5) is strongly controllable and $N_o \geq (2^{n+m}(2^{m(s^*+1)} - 1))/(2^m - 1)$. Then it is observable if and only if

$$\text{rank}\left[O_0^T, O_1^T, \ldots, O_{s^*}^T\right]^T = 2^n. \tag{20}$$

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Controllability of BCN With Multiple Time Delays

- The concept of controllability matrices
- An iterative algorithm to calculate controllability matrices
- A concise criterion for controllability
- Sufficient and necessary conditions for controllability

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I. The considered BCN

A BCN with multiple time-varying delays can be expressed as follows:

$$\begin{cases} X_1(t+1) = f_1 & (X_1(t-\tau_1(t)),\ldots,X_1(t-\tau_q(t)),\ldots, \\ & X_n(t-\tau_1(t))\ldots,X_n(t-\tau_q(t)),U_1(t), \\ & \ldots,U_m(t)) \\ & \vdots \\ X_n(t+1) = f_n & (X_1(t-\tau_1(t)),\ldots,X_1(t-\tau_q(t)),\ldots, \\ & X_n(t-\tau_1(t)),\ldots,X_n(t-\tau_q(t)),U_1(t), \\ & \ldots,U_m(t)) \end{cases} \tag{4}$$

$$x(t+1) = \hat{L}u(t)x(t-\tau_1(t))\cdots x(t-\tau_q(t)) \tag{5}$$

where $\hat{L} \in \mathscr{L}_{2^n \times 2^{m+qn}}$. Here, $\tau_i : \mathbb{Z}_{\geq 0} \to \mathcal{N}_i = \{0, 1, \ldots, T_i\}$,

Then, a standard form of BCN with multiple time-varying delays is defined by

$$x(t+1) = L(t)u(t)\mathbf{x}_t. \tag{11}$$

For every $\mathcal{N}_i$, we construct a one-to-one correspondence between $\mathcal{N}_i$ and Δ_{T_i+1}, as $\tau_i \mapsto \delta_{T_i+1}^{\tau_i+1}, \tau_i \in \mathcal{N}_i$. We define $\omega_i(t) = \delta_{T_i+1}^{\tau_i(t)+1}$, which is called the vector form of the time delay $\tau_i(t)$. Obviously, it holds that $\tau_i(t) = \mu^T \omega_i(t)$, where $\mu^T = [0\ 1\ \ldots\ T_i]$. We require the following lemma.

Lemma 1 (See [20]): For any $v_1, v_2, \ldots, v_N \in \Delta_{2^m}$ and $\omega \in \Delta_N$, it holds that

$$[v_1\ v_2\ \ldots\ v_N]\omega = G\omega v_1 v_2 \ldots v_N \tag{7}$$

where $G \in \mathcal{L}_{2^m \times N 2^{Nm}}$ is defined as follows:

$$G = [G_1\ G_2\ \ldots\ G_N] \tag{8}$$

with

$$G_j := \left(\mathbf{1}^T_{2^{(N-1)m}} \otimes I_{2^m}\right) W_{[2^{jm}, 2^{(N-j)m}]}, \quad j = 1, \ldots, N.$$

□

By Lemma 1, it holds that

$$\begin{aligned} x(t-\tau_i(t)) &= [x(t)\ x(t-1)\ \cdots\ x(t-T_i)]\omega_i(t) \\ &= \hat{G}_i \omega_i(t) x(t) x(t-1) \cdots x(t-T_i) \end{aligned}$$

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I. The definition of controllability

A standard form of BCN with multiple time-varying delays is defined by

$$x(t+1) = L(t)u(t)\mathbf{x}_t. \tag{11}$$

For a given initial state $\mathbf{x}_0 = x(t_0) \ltimes x(t_0 - 1) \ltimes \cdots \ltimes x(t_0 - \tau) \in \Delta_{2^{n(\tau+1)}}$ and a control sequence $\mathbf{u} = \{u(t)\}_{t \in \mathbb{Z}_{t \geq t_0}}$, the solution to BCN (11) is denoted by $x(t; t_0, \mathbf{x}_0, \mathbf{u})$.

Definition 1: A state $X \in \Delta_{2^n}$ is said to be k-step reachable from $\mathbf{x}_0 \in \Delta_{2^{n(\tau+1)}}$ at t_0 if there exists a control sequence $\mathbf{u} = \{u(t)\}_{t \in \mathbb{Z}_{\geq t_0}}$ such that

$$x(t_0 + k; t_0, \mathbf{x}_0, \mathbf{u}) = X. \tag{12}$$

A state $X \in \Delta_{2^n}$ is said to be reachable from $\mathbf{x}_0 \in \Delta_{2^{n(\tau+1)}}$ at t_0 if (12) holds for some positive integer.

The set of all k-step reachable states from $\mathbf{x} \in \Delta_{2^{n(\tau+1)}}$ at t is denoted by $\mathcal{R}^{(t,t+k)}(\mathbf{x})$. The set of all reachable states from $\mathbf{x}$ at t is denoted by $\mathcal{R}^t(\mathbf{x})$. For any $\mathcal{A} \subseteq \Delta_{2^{n(\tau+1)}}$ and $k \in \mathbb{Z}_{\geq 1}$, define

$$\mathcal{R}^{(t,t+k)}(\mathcal{A}) := \cup_{\mathbf{x} \in \mathcal{A}} \mathcal{R}^{(t,t+k)}(\mathbf{x}).$$

Definition 2: BCN (11) is said to be controllable at t if $\mathcal{R}^t(\mathbf{x}) = \Delta_{2^n}\ \forall \mathbf{x} \in \Delta_{2^{n(\tau+1)}}$.

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II. Controllability matrices

Definition 3 (Controllability Matrix):

1) The k-step controllability matrix $\mathbf{C}(t, t+k) \in \mathscr{B}_{2^n \times 2^{n(\tau+1)}}$ is defined as follows:

$$(\mathbf{C}(t, t+k))_{ij} = \begin{cases} 1, & \delta_{2^n}^i \in \mathcal{R}^{(t,t+k)}(\delta_{2^{n(\tau+1)}}^j) \\ 0, & \delta_{2^n}^i \notin \mathcal{R}^{(t,t+k)}(\delta_{2^{n(\tau+1)}}^j). \end{cases} \quad (13)$$

2) The controllability matrix $\mathbf{C}_t \in \mathscr{B}_{2^n \times 2^{n(\tau+1)}}$ is defined as follows:

$$(\mathbf{C}_t)_{ij} = \begin{cases} 1, & \delta_{2^n}^i \in \mathcal{R}^t(\delta_{2^{n(\tau+1)}}^j) \\ 0, & \delta_{2^n}^i \notin \mathcal{R}^t(\delta_{2^{n(\tau+1)}}^j). \end{cases} \quad (14)$$

Controllability matrix

By the definitions, a BCN is controllable at t if and only if all components of $\mathbf{C}_t$ are equal to 1.

Lemma 2: The following statements hold.

1) For any logical vector $X \in \Delta_{2^{n(\tau+1)}}$, it holds that

$$\mathcal{R}^{(t,t+k)}(X) = \mathscr{S}(\mathbf{C}(t, t+k)X).$$

2) For any Boolean vector $Y \in \mathscr{B}_{2^{n(\tau+1)} \times 1}$ and any positive integer k, it holds that

$$\mathcal{R}^{(t,t+k)}[\mathscr{S}(Y)] = \mathscr{S}[\mathbf{C}(t, t+k) \ltimes_{\mathscr{B}} Y]. \quad (15)$$

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II. Controllability matrices

Proposition 1: The k-step controllability matrices for BCN (11) are given by

$$\mathbf{C}(t,t+1) = L(t) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}$$

$$\mathbf{C}(t,t+k) = \mathbf{C}(t+1,t+k) \ltimes_{\mathscr{B}} \mathbf{C}(t,t+1)\Gamma$$

$$k = 2,3,\ldots$$

where $\Gamma := W_{[2^{n\tau},2^{n(\tau+1)}]} M_{r,2^{n\tau}}$.

Proof: For any state $\mathbf{x} \in \Delta_{2^{n(\tau+1)}}$, we define a Boolean vector $\mathbf{r}^{(t,t+1)}(\mathbf{x}) = (L(t) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})\mathbf{x}$. By (11), it holds that

$$\mathscr{S}\left(\mathbf{r}^{(t,t+1)}(\mathbf{x})\right) = \mathcal{R}^{(t,t+1)}(\mathbf{x}).$$

Thus, we have that

$$\mathbf{C}(t,t+1) = \left[\mathbf{r}^{(t,t+1)}\left(\delta^1_{2^{n(\tau+1)}}\right),\ldots,\mathbf{r}^{(t,t+1)}\left(\delta^{2^{n(\tau+1)}}_{2^{n(\tau+1)}}\right)\right]$$

$$= L(t) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}.$$

For any $x_d \in \Delta_{2^n}$, $x_d \in \mathcal{R}^{(t,t+k)}(\mathbf{x})$ if and only if there exists $z \in \mathcal{R}^{(t,t+1)}$ such that

$$x_d \in \mathcal{R}^{(t+1,t+k)}[zx(t)\cdots x(t-\tau+1)].$$

Then, for $k \geq 2$, it holds that

$$\mathcal{R}^{(t,t+k)}(\mathbf{x}) = \bigcup_{z\in\mathcal{R}^{(t,t+1)}(\mathbf{x})} \mathcal{R}^{(t+1,t+k)}[zx(t)\cdots x(t-\tau+1)]$$

$$= \mathcal{R}^{(t+1,t+k)}\{\mathscr{S}[\mathbf{C}(t,t+1)\mathbf{x}x(t)\cdots x(t-\tau+1)]\}.$$

Note that

$$\begin{aligned}\mathbf{x}x(t)\cdots x(t-\tau+1) &= x(t)x(t-1)\cdots x(t-\tau)x(t)\cdots x(t-\tau+1)\\ &= W_{[2^{n\tau},2^{n(\tau+1)}]}M_{r,2^{n\tau}}x(t)\cdots x(t-\tau+1)x(t-\tau)\\ &= \Gamma\mathbf{x}.\end{aligned}$$

Thus, by Lemma 2, it holds that

$$\begin{aligned}\mathcal{R}^{(t,t+k)}(\mathbf{x}) &= \mathcal{R}^{(t+1,t+k)}[\mathscr{S}(\mathbf{C}(t,t+1)\Gamma\mathbf{x})]\\ &= \mathscr{S}[\mathbf{C}(t+1,t+k) \ltimes_{\mathscr{B}} (\mathbf{C}(t,t+1)\Gamma)\mathbf{x}]\end{aligned}$$

which implies that $\mathbf{C}(t,t+k) = \mathbf{C}(t+1,t+k) \ltimes_{\mathscr{B}} \mathbf{C}(t,t+1)\Gamma$. This completes the proof. □

II. Controllability matrices

Corollary 1: For all $k \geq 2$, the k-step controllability matrices for BCN (11) can be calculated as follows:

$$\mathbf{C}(t, t+k) = L(t+k-1) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \ltimes_{\mathscr{B}} \left\{ \ltimes_{\mathscr{B}}{}_{i=2}^{k} [(L(t+k-i) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})\Gamma] \right\}.$$

According to the definition of controllability matrices and Proposition 1, the controllability matrix of BCN (11) is given by

$$\mathbf{C}_t = (\mathscr{B}) \sum_{k=1}^{+\infty} \mathbf{C}(t, t+k). \quad (16)$$

Proposition 2: BCN (11) is controllable at a given initial time instant t if and only if there exists an integer i such that all components of $\mathbf{Q}_i(t)$ are equal to 1, where $\mathbf{Q}_i(t) \in \mathscr{B}_{2^n \times 2^{n(\tau+1)}}$ is defined as follows:

$$\mathbf{Q}_i(t) := (\mathscr{B}) \sum_{k=1}^{i} \mathbf{C}(t, t+k). \quad (17)$$

Proof: According to Proposition 1, it holds that

$$\begin{aligned}
\mathbf{C}(t, t+k) &= \mathbf{C}(t+1, t+k) \ltimes_{\mathscr{B}} (\mathbf{C}(t, t+1)\Gamma) \\
&= \mathbf{C}(t+2, t+k) \ltimes_{\mathscr{B}} [\mathbf{C}(t+1, t+2)\Gamma] \\
&\quad \ltimes_{\mathscr{B}} [\mathbf{C}(t, t+1)\Gamma] \\
&= \cdots \\
&= \mathbf{C}(t+k-1, t+k) \ltimes_{\mathscr{B}} \\
&\quad \left\{ \ltimes_{\mathscr{B}}{}_{i=2}^{k} [\mathbf{C}(t+k-i, t+k-i+1)\Gamma] \right\}.
\end{aligned}$$

It is easy to check that for any $1 \leq i \leq k$, it holds that

$$\mathbf{C}(t+k-i, t+k-i+1) = L(t+k-i) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}.$$

Thus,

$$\mathbf{C}(t, t+k) = L(t+k-1) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \ltimes_{\mathscr{B}} \left\{ \ltimes_{\mathscr{B}}{}_{i=2}^{k} [(L(t+k-i) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m})\Gamma] \right\}.$$

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III. BCNs With Multiple Periodic Delays

Lemma 3: The following claims hold.

1) The k-step controllability matrix $\mathbf{C}(t, t+k)$ for BCN (19) is T-periodic with respect to the variable t; that is

$$\mathbf{C}(t+T, t+T+k) = \mathbf{C}(t, t+k) \quad \forall t, \forall k.$$

2) The k-step controllability matrix $\mathbf{C}(t, t+k)$ for BCN (19) is pseudo T-periodic with respect to the variable k, in the sense that

$$\mathbf{C}(t, t+T+k) = \mathbf{C}(t, t+k) \ltimes_{\mathscr{B}} \mathbf{M}(t) \quad \forall t, \forall k$$

where $\mathbf{M}(t) \in \mathscr{B}_{2^{n(\tau+1)} \times 2^{n(\tau+1)}}$ is a Boolean matrix-valued function defined by

$$\mathbf{M}(t) := \ltimes_{\mathscr{B}}{}_{j=1}^{T} \left[(L(t+T-j) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}) \Gamma \right]. \quad (21)$$

3) The controllability matrix $\mathbf{C}_t$ is T-periodic; that is, $\mathbf{C}_{t+T} = \mathbf{C}_t \quad \forall t$.

Proof: The T-periodicity of $\mathbf{C}(t, t+k)$ with respect to t is an immediate outcome of Corollary 1, because $L(t)$ is T-periodic. Thus, we only need to prove claim 2). By Corollary 1 and (20), it holds that

$$\begin{aligned}
\mathbf{C}(t, t+T+k) &= L(t+T+k-1) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \ltimes_{\mathscr{B}} \\
&\quad \left[\ltimes_{\mathscr{B}}{}_{i=2}^{T+k} (L(t+T+k-i) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \Gamma) \right] \\
&= L(t+T+k-1) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \ltimes_{\mathscr{B}} \\
&\quad \left[\ltimes_{\mathscr{B}}{}_{i=2}^{k} (L(t+T+k-i) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \Gamma) \right] \\
&\quad \ltimes_{\mathscr{B}} \left[\ltimes_{\mathscr{B}}{}_{j=1}^{T} (L(t+T-j) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \Gamma) \right] \\
&= L(t+k-1) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \ltimes_{\mathscr{B}} \\
&\quad \left[\ltimes_{\mathscr{B}}{}_{i=2}^{k} (L(t+k-i) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \Gamma) \right] \\
&\quad \ltimes_{\mathscr{B}} \left[\ltimes_{\mathscr{B}}{}_{j=1}^{T} (L(t+T-j) \ltimes_{\mathscr{B}} \mathbf{1}_{2^m} \Gamma) \right] \\
&= \mathbf{C}(t, t+k) \ltimes_{\mathscr{B}} \mathbf{M}(t).
\end{aligned}$$

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III. BCNs With Multiple Periodic Delays

Proposition 3: The controllability matrix of BCN (5)–(18) is given by

$$\mathbf{C}_t = \left[(\mathscr{B})\sum_{w=1}^{T}\mathbf{C}(t,t+w)\right] \ltimes_{\mathscr{B}} \left[(\mathscr{B})\sum_{s=0}^{2^{2n(\tau+1)}}\mathbf{M}^{(s)}(t)\right] \tag{22}$$

where τ is defined in (6), and $\mathbf{M}^{(0)}(t) := I_{2^{n(\tau+1)}}$.

Proof: By Lemma 3, it holds for any positive integer m that

$$\begin{aligned}\mathbf{C}(t,t+mT+k) &= \mathbf{C}(t,t+(m-1)T+k) \ltimes_{\mathscr{B}} \mathbf{M}^{(1)}(t)\\ &= \mathbf{C}(t,t+(m-2)T+k) \ltimes_{\mathscr{B}} \mathbf{M}^{(2)}(t)\\ &= \cdots\\ &= \mathbf{C}(t,t+k) \ltimes_{\mathscr{B}} \mathbf{M}^{(m)}(t).\end{aligned}$$

In particular, the aforementioned formula also holds for $m = 0$ with $\mathbf{M}^{(0)}(t) := I_{2^{n(\tau+1)}}$. Thus, by (6), it holds that

$$\begin{aligned}\mathbf{C}_t &= (\mathscr{B})\sum_{k=1}^{+\infty}\mathbf{C}(t,t+k)\\ &= (\mathscr{B})\sum_{s=0}^{+\infty}(\mathscr{B})\sum_{k=sT+1}^{(s+1)T}\mathbf{C}(t,t+k)\\ &= (\mathscr{B})\sum_{s=0}^{+\infty}(\mathscr{B})\sum_{w=1}^{T}\mathbf{C}(t,t+sT+w)\\ &= (\mathscr{B})\sum_{s=0}^{+\infty}(\mathscr{B})\sum_{w=1}^{T}\mathbf{C}(t,t+w) \ltimes_{\mathscr{B}} \mathbf{M}^{(s)}(t)\\ &= \left[(\mathscr{B})\sum_{w=1}^{T}\mathbf{C}(t,t+w)\right] \ltimes_{\mathscr{B}} \left[(\mathscr{B})\sum_{s=0}^{+\infty}\mathbf{M}^{(s)}(t)\right].\end{aligned}$$

Because $\mathbf{M}(t)$ is a $2^{n(\tau+1)} \times 2^{n(\tau+1)}$ Boolean matrix, it holds that

$$(\mathscr{B})\sum_{s=0}^{+\infty}\mathbf{M}^{(s)}(t) = (\mathscr{B})\sum_{s=0}^{2^{2n(\tau+1)}}\mathbf{M}^{(s)}(t)$$

and, thus, (22) follows. □

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III. BCNs With Multiple Periodic Delays

Proposition 4: BCN (19) with periodic delays is controllable at $t_0 \geq 0$ if and only if it is controllable at $t = 0$; that is, all components of $\mathbf{C}_0$ are equal to 1.

Proof: We only need to prove the sufficiency of this condition. Suppose that all components of $\mathbf{C}_0$ are equal to 1. For any given initial time instant t_0, let $m \in \mathbb{Z}_{\geq 0}$ and $0 \leq k \leq T-1$ be integers such that $t_0 = mT + k$. Let $\mathbf{x}_{t_0} = x(t_0) \ltimes x(t_0 - 1) \ltimes \cdots \ltimes x(t_0 - \tau) \in \Delta_{2^{n(\tau+1)}}$ be any initial state and $\mathbf{u}_1 = \{u_1(t)\}_{t_0 \leq t \leq (m+1)T-1}$ be any segment of control sequence. Suppose that under this control sequence, the system evolves to $\mathbf{x}_{(m+1)T} = x((m+1)T) \ltimes x((m+1)T - 1) \ltimes \cdots \ltimes x((m+1)T - \tau) \in \Delta_{2^{n(\tau+1)}}$ at the time instant $t = (m+1)T$. By the T-periodicity of $\mathbf{C}_t$ proved in Lemma 3, we have that $\mathbf{C}_{(m+1)T} = \mathbf{C}_0$, whose components are all equal to 1. Thus, for any target state $X \in \Delta_{2^n}$ there exists a control sequence $\mathbf{u}_2 = \{u_2(t)\}_{(m+1)T \leq t \leq K-1}$ such that

$$x(K; (m+1)T, \mathbf{x}_{(m+1)T}, \mathbf{u}_2) = X.$$

Define a control sequence $\mathbf{u} = \{u(t)\}_{t_0 \leq t \leq K-1}$ as follows:

$$u(t) = \begin{cases} u_1(t), & t_0 \leq t \leq (m+1)T - 1 \\ u_2(t), & (m+1)T \leq t \leq K - 1. \end{cases}$$

Then, $x(K; t_0, \mathbf{x}_{t_0}, \mathbf{u}) = X$. Because t_0, $\mathbf{x}_{t_0}$, and X are chosen arbitrarily, we have that $\mathcal{R}_t(\mathbf{x}) = \Delta_{2^n}$ $\forall \mathbf{x} \in \Delta_{2^{n(\tau+1)}}$ $\forall t \in \mathbb{Z}_{\geq 0}$. □

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III. BCNs With Multiple Periodic Delays

Suppose that the time delays of BCN (5) are constant; that is, $\tau_i(t) \equiv a_i$, where $a_i \in \mathbb{Z}_{\geq 0}$ are non-negative integers. Then, the BCN can be expressed as follows:

$$x(t+1) = Lu(t)\mathbf{x}_t. \quad (23)$$

The set of all k-step reachable states from x_0 is denoted by $R^{(k)}(x_0)$. The set of all reachable states from x_0 is denoted by $R(x_0)$. The k-step controllability matrix $C_k \in B_{2^n \times 2^{n(\tau+1)}}$ is defined as follows:

$$(\mathbf{C}_k)_{ij} = \begin{cases} 1, & \delta_{2^n}^i \in \mathcal{R}^{(k)}(\delta_{2^{n(\tau+1)}}^j) \\ 0, & \delta_{2^n}^i \notin \mathcal{R}^{(k)}(\delta_{2^{n(\tau+1)}}^j) \end{cases}$$

And the controllability matrix $C \in B_{2^n \times 2^{n(\tau+1)}}$ is defined as follows:

$$(\mathbf{C})_{ij} = \begin{cases} 1, & \delta_{2^n}^i \in \mathcal{R}(\delta_{2^{n(\tau+1)}}^j) \\ 0, & \delta_{2^n}^i \notin \mathcal{R}(\delta_{2^{n(\tau+1)}}^j). \end{cases}$$

Proposition 5: The k-step controllability matrices for BCN (23) are given by

$$\mathbf{C}_1 = L \ltimes_{\mathscr{B}} \mathbf{1}_{2^m}$$
$$\mathbf{C}_k = \mathbf{C}_{k-1} \ltimes_{\mathscr{B}} (\mathbf{C}_1\Gamma) = \mathbf{C}_1 \ltimes_{\mathscr{B}} (\mathbf{C}_1\Gamma)^{(k-1)}$$
$$k = 2, 3, \ldots$$

where $\Gamma = W_{[2^{n\tau}, 2^{n(\tau+1)}]} M_{r, 2^{n\tau}}$. □

Proposition 6: The controllability matrix for BCN (23) is given by

$$\mathbf{C} = (\mathscr{B}) \sum_{k=1}^{2^{n(\tau+1)}} \mathbf{C}_k = \mathbf{C}_1 \ltimes_{\mathscr{B}} \left[(\mathscr{B}) \sum_{k=1}^{2^{n(\tau+1)}} (\mathbf{C}_1\Gamma)^{(k-1)} \right]$$

where $(\mathbf{C}_1\Gamma)^{(0)} := I_{2^{n(\tau+1)}}$. In addition, BCN (23) is controllable if and only if all components of $\mathbf{C}$ are equal to 1; that is

$$\mathrm{Col}_i(\mathbf{C}) = \mathbf{1}_{2^n}, \quad \forall i = 1, 2, \ldots, 2^{n(\tau+1)}. \quad (26)$$

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IV. BCNs With Multiple Constant Delays

Based on the definitions of controllability matrices, a state x_d is k-step reachable from $\mathbf{x}_0 = x_0 \ltimes \cdots \ltimes x_{-\tau}$ if and only if

$$x_d^T \mathbf{C}_k \mathbf{x}_0 = 1. \tag{27}$$

A path

$$\mathbf{x}_0 \to x_1 \to x_2 \to \cdots \to x_{k-1} \to x_k = x_d \tag{28}$$

is called an admissible path if there is a control sequence $\mathbf{u}$ such that

$$x(j; \mathbf{x}_0, \mathbf{u}) = x_j \quad \forall j \in \{1, 2, \ldots, k\}.$$

Lemma 4: Suppose that $X \in \Delta_n$, $Y \in \Delta_m$, and $B \in \mathscr{B}_{m \times n}$. Then

$$Y \in \mathscr{S}(BX) \tag{29}$$

if and only if

$$X \in \mathscr{S}^T(Y^T B). \tag{30}$$

Proposition 7: Consider BCN (9). A path (28) is admissible if and only if

$$x_i \in \mathcal{S}_{i-1} \quad \forall i \in \{1, 2, \ldots, k-1\}$$

where

$$\mathcal{S}_{i-1} := \mathscr{S}[\mathbf{C}_1 \mathbf{x}_{i-1} \wedge (x_d^T \mathbf{C}_{k-i} W_{[2^{n\tau}, 2^n]} D_{[1, 2^{n\tau}, 2^n]} \mathbf{x}_{i-1})^T].$$

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V. Example

Consider a BCN with a constant time delay

$$x(t+1) = Lu(t)x(t)x(t-1)x(t-2) \quad (37)$$

with $x \in \Delta_4$, $u \in \Delta_2$, and

$$L = \delta_4[32142143\ 12233143\ 42442331\ 42321213$$
$$22134142\ 11412244\ 14244343\ 22121213$$
$$24442144\ 32334444\ 12432331\ 42221223$$
$$22442142\ 13333143\ 44432233\ 24321211].$$

Because $\tau = 2$, the space of initial states is Δ_{64}. By Proposition 5, we can calculate all k-step controllability matrices $\mathbf{C}_i$, with $1 \leq i \leq 64$. For instance, the three-step controllability matrix $\mathbf{C}_3$ is given by

$$\mathbf{C}_3 = \begin{bmatrix} 0\,1\,1\,1\,0\,1\,1\,1\,1\,0 \cdots\cdots 0\,0\,1\,1\,0\,1\,0\,0\,0\,1 \\ 1\,1\,1\,0\,1\,1\,0\,0\,1\,1 \cdots\cdots 1\,1\,0\,0\,1\,0\,0\,1\,0\,1 \\ 0\,1\,1\,1\,0\,0\,0\,0\,1\,0 \cdots\cdots 1\,1\,1\,1\,1\,1\,1\,0\,1\,1 \\ 1\,1\,1\,1\,1\,0\,1\,0\,1\,1 \cdots\cdots 0\,0\,1\,1\,1\,1\,1\,0\,1\,1 \end{bmatrix}.$$

One sees that $\mathrm{Col}_4(\mathbf{C}_3) = [1\ 0\ 1\ 1]^T$, and so the three-step reachable set from $\delta_{64}^4 = \delta_4^1 \ltimes \delta_4^1 \ltimes \delta_4^4$ is $\mathcal{R}^{(3)}(\delta_{64}^4) = \{\delta_4^1, \delta_4^3, \delta_4^4\}$. In addition, because the $(2,4)$-entry of $\mathbf{C}_3$ is zero, there exists no control sequence that drives the network from δ_{64}^4 to δ_4^2 in exactly three steps. The controllability matrix $\mathbf{C}$ of this BCN can be calculated by using Proposition 6, which is omitted here owing to restrictions on space. It is easy to check that

$$(\mathbf{C})_{ij} = 1 \quad \forall 1 \leq i \leq 4, \forall 1 \leq j \leq 64.$$

Thus, BCN (37) is controllable.

In the following, we set the initial and destination states as $\mathbf{x}_0 = x_0 \ltimes x_{-1} \ltimes x_{-2} = \delta_4^4 \ltimes \delta_4^1 \ltimes \delta_4^2 = \delta_{64}^{50}$ and $x_d = \delta_4^2$, respectively. We aim to design a control sequence to drive the BCN from $\mathbf{x}_0$ to x_d in the fewest steps. It can be shown that $(\delta_4^2)^T \mathbf{C}_i \mathbf{x}_0 = 0$ for $i = 1, 2, 3, 4$, and $(\delta_4^2)^T \mathbf{C}_5 \mathbf{x}_0 = 1$. This implies that at least five steps are required to steer the system from $\mathbf{x}_0$ to x_d. An admissible path from $\mathbf{x}_0$ to x_d can be obtained. More precisely, by Proposition 7 it holds that

$$x_1 \in \mathcal{S}_0 = \mathscr{S}[\mathbf{C}_1 \mathbf{x}_0 \wedge (x_d^T \mathbf{C}_4 W_{[16,4]} D_{[1,16,4]} \mathbf{x}_0)^T]$$
$$= \mathscr{S}[(0\,0\,0\,1)^T \wedge (1\,0\,0\,1)^T] = \{\delta_4^4\}.$$

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V. Example

Consider a BCN with a constant time delay

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with $x \in \Delta_4$, $u \in \Delta_2$, and

$$L = \delta_4[32142143\ 12233143\ 42442331\ 42321213$$
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Because $\tau = 2$, the space of initial states is Δ_{64}. By Proposition 5, we can calculate all k-step controllability matrices $\mathbf{C}_i$, with $1 \le i \le 64$. For instance, the three-step controllability matrix $\mathbf{C}_3$ is given by

$$\mathbf{C}_3 = \begin{bmatrix} 0\,1\,1\,1\,0\,1\,1\,1\,1\,0 \cdots\cdots 0\,0\,1\,1\,0\,1\,0\,0\,0\,1 \\ 1\,1\,1\,0\,1\,1\,0\,0\,1\,1 \cdots\cdots 1\,1\,0\,0\,1\,0\,0\,1\,0\,1 \\ 0\,1\,1\,1\,0\,0\,0\,0\,1\,0 \cdots\cdots 1\,1\,1\,1\,1\,1\,1\,0\,1\,1 \\ 1\,1\,1\,1\,1\,0\,1\,0\,1\,1 \cdots\cdots 0\,0\,1\,1\,1\,1\,1\,0\,1\,1 \end{bmatrix}.$$

Hence, $x_1 = \delta_4^4$, and

$$x_2 \in \mathcal{S}_1 = \mathscr{S}[\mathbf{C}_1\mathbf{x}_1 \wedge (x_d^T\mathbf{C}_3W_{[16,4]}D_{[1,16,4]}\mathbf{x}_1)^T]$$
$$= \{\delta_4^4\}.$$

Thus, $x_2 = \delta_4^4 \in \mathcal{S}_1$. By following the same argument, it holds that $\mathcal{S}_2 = \{\delta_4^3, \delta_4^4\}$. If we choose $x_3 = \delta_4^3$, then $\mathcal{S}_3 = \{\delta_4^4\}$. That is, $x_4 = \delta_4^4$. Finally, an admissible path is given by

$$\underbrace{\delta_4^2 \rightarrow \delta_4^1 \rightarrow \delta_4^4}_{\mathbf{x}_0 = \delta_4^4 \ltimes \delta_4^1 \ltimes \delta_4^2 = \delta_{64}^{50}} \xrightarrow{u_0} \delta_4^4 \xrightarrow{u_1} \delta_4^1 \xrightarrow{u_2} \delta_4^3 \xrightarrow{u_3} \delta_4^4 \xrightarrow{u_4} \delta_4^2. \quad (38)$$

In (38), the u_i's labeling the arrows represent the corresponding control efforts, which can be calculated by using (33). For instance, we have that

$$u_0 \in \mathscr{S}^T[x_0^T LW_{[2^6,2]}\mathbf{x}_0]$$
$$= \mathscr{S}(\mathbf{1}_2^T) = \{\delta_2^1, \delta_2^2\}.$$

Similarly, $u_1 \in \{\delta_2^1, \delta_2^2\}$, $u_2 \in \{\delta_2^1\}$, $u_3 \in \{\delta_2^2\}$, and $u_4 \in \{\delta_2^1, \delta_2^2\}$.

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End

Stability of Boolean Networks With Delays Using Pinning Control

I. Problem Formulation

A BN with a set of Boolean variables $x_1, \ldots, x_n$ and time delays is described as

$$x_i(t+1) = f_i(x_1(t), \ldots, x_n(t), x_1(t-1), \ldots, x_n(t-1), \ldots, x_1(t-\tau), \ldots, x_n(t-\tau)) \quad i = 1, 2, \ldots, n \tag{1}$$

named as higher order BNs or $(\tau+1)th$-order BNs

Definition 1

A Boolean control network with time delays (1) is **globally stabilized** to the fixed point $\delta_{2^n}^a$, if for arbitrary initial state sequence $x(0), x(-1), \ldots, x(-\tau) \in \Delta_{2^n}$, there exist control inputs and $T \in Z_+$ such that $x(t) = \delta_{2^n}^a$, for every $t \geq T$.

Li F , et al. Controllability of Boolean control networks with time delays in states [J], 2011

Denote by $E_1(\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}})$ the set consisting of all the states $\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q}$ that can be steered to $\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}$ in one step. That is $E_1(\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}) = \{\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q} \in \Delta_{2^{n(\tau+1)}} : x(1;\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q}) = \delta_{2^n}^{a_1}\}$. In addition, $E_k(\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}})$ the set consisting of all the states $\delta_{2^n}^{a_{k+1}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k}$ that can be steered to $\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}$ in k steps. In order to calculate these sets, we provide the following Lemma.

Lemma 3.1: Let $\alpha_1,\ldots,\alpha_{2^{n(\tau+1)}} \in \{1,2,\ldots,2^n\}$ such that the transition matrix L of the BNs (1) is

$$L = \delta_{2^n}[\alpha_1,\ldots,\alpha_{2^{n(\tau+1)}}].$$

Then, for every $1 \le a_1,\ldots,\alpha_{2^{n(\tau+1)}} \le 2^n$

1) $E_1(\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}) = \{\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q} : 1 \le q \le 2^n, \alpha_{(a_2-1)2^{n\tau}+(a_3-1)2^{n(\tau-1)}+\cdots+q} = a_1\}$.
2) For all of the $\delta_{2^n}^{a_{k+1}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k} \in E_k(\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}})$, we have $E_{k+1}(\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}) = E_1(\delta_{2^n}^{a_{k+1}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k})$.

Proof :

1) It can be noted that for $x(t)x(t-1)\cdots x(t-\tau) = \delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q}$, we have

$$\begin{aligned} x(t+1) &= Lx(t)x(t-1)\cdots x(t-\tau) \\ &= L\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q} \\ &= \delta_{2^n}^{\alpha_{(a_2-1)2^{n\tau}+(a_3-1)2^{n(\tau-1)}+\cdots+q}} \\ &= \delta_{2^n}^{a_1}. \end{aligned}$$

Thus, $\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q}$ can be steered to $\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}$ in one step.

2) The physical meaning of $E_1(\delta_{2^n}^{a_{k+1}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k})$ is that the state $\delta_{2^n}^{a_{k+2}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k}\delta_{2^n}^{q_{k+1}}$ can reach $\delta_{2^n}^{a_{k+1}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k}$ in one step.

$\delta_{2^n}^{a_{k+1}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k} \in E_k(\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}})$ implies that $\delta_{2^n}^{a_{k+1}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k}$ can reach $\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}$ in k steps. That is, $\delta_{2^n}^{a_{k+2}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k}\delta_{2^n}^{q_{k+1}}$ can reach $\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}$ in $k+1$ steps, i.e., $E_{k+1}(\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}) = E_1(\delta_{2^n}^{a_{k+1}}\cdots\delta_{2^n}^{a_{\tau+1}}\delta_{2^n}^{q_1}\cdots\delta_{2^n}^{q_k})$.

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II. Algorithm & Proposition

Algorithm 3.1:

Step 1. Change the $[(a-1)2^{n\tau}+(a-1)2^{n(\tau-1)}+\cdots+a]$th column of L to $\delta_{2^n}^a$.

Step 2. Calculate $\mathbf{E}(\underbrace{\delta_{2^n}^a\cdots\delta_{2^n}^a}_{\tau+1}) := \bigcup_{k=1}^{2^{n(\tau+1)}} E_k(\underbrace{\delta_{2^n}^a\cdots\delta_{2^n}^a}_{\tau+1})$.

Step 3. Find a $\delta_{2^n}^{a_1}\cdots\delta_{2^n}^{a_{\tau+1}} \notin \mathbf{E}(\underbrace{\delta_{2^n}^a\cdots\delta_{2^n}^a}_{\tau+1})$ and we can change the column $\mathrm{Col}_{[(a_1-1)2^{n\tau}+(a_2-1)2^{n(\tau-1)}+\cdots+a_{\tau+1}]}(L)$ into $\delta_{2^n}^\alpha$ such that $\delta_{2^n}^{a_1}\cdots\delta_{2^n}^{a_{\tau+1}} \in E_1(\delta_{2^n}^\alpha\delta_{2^n}^{a_1}\cdots\delta_{2^n}^{a_\tau})$ and $\delta_{2^n}^\alpha\delta_{2^n}^{a_1}\cdots\delta_{2^n}^{a_\tau} \in \mathbf{E}(\underbrace{\delta_{2^n}^a\cdots\delta_{2^n}^a}_{\tau+1})$.

By doing this, L is changed to L' and (1) is globally stabilized to the fixed point $\delta_{2^n}^a$.

Proposition 3.1: Let $x^* = \delta_{2^n}^a$. Suppose that L is changed into L' according to Algorithm 3.1, then the BNs with time delays (1) with the transition matrix L' are globally stabilized to x^*.

Proof: For any initial states $\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}} \in \Delta_{2^{n(\tau+1)}}$, then $\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}} \in \mathbf{E}(\underbrace{\delta_{2^n}^a\cdots\delta_{2^n}^a}_{\tau+1})$ according to Algorithm 3.1. Hence, $\delta_{2^n}^{a_1}\delta_{2^n}^{a_2}\cdots\delta_{2^n}^{a_{\tau+1}}$ can reach $\underbrace{\delta_{2^n}^a\cdots\delta_{2^n}^a}_{\tau+1}$ in the most k steps, where $k = 2^{n(\tau+1)}$.

It can be noted that

$$\begin{aligned} x(k+1) &= L'x(k)x(k-1)\cdots x(k-\tau) \\ &= L'\delta_{2^n}^a\cdots\delta_{2^n}^a\delta_{2^n}^a \\ &= \mathrm{Col}_{[(a-1)2^{n\tau}+(a-2)2^{n(\tau-1)}+\cdots+a]}(L') \\ &= \delta_{2^n}^a \end{aligned}$$

which yields $x(t) = \delta_{2^n}^a$, for $t \geq k$. That is, the BNs with time delays (1) starting from any initial states can reach $\delta_{2^n}^a$ and stay at $\delta_{2^n}^a$ forever. We can draw the conclusion due to the arbitrariness of initial states. ■

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III. The design procedure

Assume that the transition matrix L of BNs (1) is changed to L according to Algorithm 3.1. Without loss of generality, we assume that the 1st,…, mth columns of L alter and assume that 1st,…, mth columns of $M_1, \dots, M_k$ alter. We assume that $M_1, \dots, M_k$ alter to $M'_1, \dots, M'_k$.

Suppose that $f_i(x_1(t), \cdots, x_n(t-\tau)))$ be changed to

$$F_i\big(u_i, x_1(t), \cdots x_n(t-\tau)\big) = u_i\big(x_1(t), \cdots x_n(t-\tau)\big) \oplus_i f_i\big(x_1(t), \cdots x_n(t-\tau)\big)$$

where $i = 1, \dots, k$, $\oplus_i$ are logical functions, and u_i are state feedback control inputs.

It can be verified that

$$F_i\big(u_i, x_1(t), \cdots x_n(t-\tau)\big) = M_{\oplus_i}\bar{M}_i\big(I_{2^{n(\tau+1)}} \otimes M_1\big)\Phi_{n(\tau+1)}x_1(t), \cdots x_n(t-\tau)$$

$$\begin{cases} M_{\oplus_1}\bar{M}_1(I_{2^{n(\tau+1)}} \otimes M_1)\Phi_{n(\tau+1)} = M'_1 \\ \vdots \\ M_{\oplus_k}\bar{M}_k(I_{2^{n(\tau+1)}} \otimes M_k)\Phi_{n(\tau+1)} = M'_k \end{cases} \quad (7)$$

Proposition 3.2: Equation (7) is solvable.

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IV. Algorithm of pinning control for BNs (1)

The algorithm to design pinning control

Algorithm 3.2:

1) Change the columns of the transition matrix L of (1) by using Algorithm 3.1.
2) Calculate the new structure matrices. Without loss of generality, we assume that the 1st, ..., mth columns of $M_1, \ldots, M_k$ alter to $M_1' \cdots, M_k'$.

3) Suppose that $f_i(x_1(t), \cdots, x_n(t-\tau)))$ be changed to

$$F_i\big(u_i, x_1(t), \cdots x_n(t-\tau)\big)$$
$$= u_i\big(x_1(t), \cdots x_n(t-\tau)\big) \oplus_i f_i\big(x_1(t), \cdots x_n(t-\tau)\big)$$

where $i = 1, \cdots k$. Solve $M_{\oplus_i}, \bar{M}_i$ from (7) by using proposition 3.2. Then, one can obtain the logical functions $\oplus_i, u_i$. Hence, the BNs (1) are **globally stabilizable** to the fixed point $\delta_{2^n}^a$.

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V. Example

Consider the following biochemical network of coupled oscillations in the cell cycle

$$\begin{cases} A(t+1) = f_1 = \neg(A(t-2) \wedge B(t-1)) \\ B(t+1) = f_2 = \neg(A(t-1) \wedge B(t-2)). \end{cases} \quad (11)$$

We want to stabilize the BNs (11) to δ_4^1.

By calculation, we have

$$\begin{aligned} L = \delta_4[&4,3,2,1,2,1,2,1,3,3,1,1,1,1,1,1,\\ &4,3,2,1,2,1,2,1,3,3,1,1,1,1,1,1,\\ &4,3,2,1,2,1,2,1,3,3,1,1,1,1,1,1,\\ &4,3,2,1,2,1,2,1,3,3,1,1,1,1,1,1]. \end{aligned}$$

Using Algorithm 3.1, we can calculate that

$\delta_4^i \delta_4^j \delta_4^k \in E(\delta_4^1 \delta_4^1 \delta_4^1)$ for $i,\ j,\ k =\ 1,2,3,4$.

Hence, the transition matrix L of (11) can be changed into L', that is,

$$\begin{aligned} L' = \delta_4[&1,1,2,1,2,1,2,1,3,3,1,1,1,1,1,1,\\ &4,3,2,1,2,1,2,1,3,3,1,1,1,1,1,1,\\ &4,3,2,1,2,1,3,1,3,3,1,1,1,1,1,1,\\ &4,3,2,1,2,1,2,1,3,3,1,1,1,1,1,1]. \end{aligned}$$

Using Algorithm 3.2, we can calculate that

$$M_{\oplus_1} = M_{\oplus_2} = \delta_2[1,2,2,1]$$

$$\begin{aligned} \bar{M}_1 = \delta_2[&2,2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\ &1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\ &1,1,1,1,1,1,2,1,1,1,1,1,1,1,1,1,1,1,\\ &1,1,1,1,1,1,1,1,1,1,1,1,1,1] \end{aligned}$$

$$\begin{aligned} \bar{M}_1 = \delta_2[&2,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\ &1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\ &1,1,1,1,1,1,2,1,1,1,1,1,1,1,1,\\ &1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]. \end{aligned}$$

Then, we have

$$\begin{aligned} u_1(t) = &[A(t) \wedge (\neg B(t) \vee \neg A(t-1) \vee \neg B(t-1) \vee \neg A(t-2))]\\ &\vee [\neg A(t) \wedge (\neg B(t) \vee \neg A(t-1)\\ &\vee B(t-1) \vee A(t-2) \vee \neg B(t-2)].\\ u_2(t) = &[A(t) \wedge (\neg B(t) \vee \neg A(t-1)\\ &\vee \neg B(t-1) \vee \neg(A(t-2) \vee B(t-2))]\\ &\vee [\neg A(t) \wedge (\neg B(t) \vee \neg A(t-1)\\ &\vee B(t-1) \vee A(t-2) \vee \neg B(t-2)]. \end{aligned}$$

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End

Synchronization in an Array of Output-Coupled BNs With Time Delay

I. Problem Formulation

We consider the following two kinds of arrays of M delayed coupled BNs, with each BN being an N-nodes system:

$$\begin{cases} X_j^i(t+1) = f_j^i(X_j^1(t-\tau), X_j^2(t-\tau), \ldots, X_j^N(t-\tau) \\ \qquad\qquad y_1(t-\tau), y_2(t-\tau), \ldots, y_M(t-\tau)) \\ y_j(t) = g_j(X_j^1(t), \ldots, X_j^N(t)) \end{cases} \quad (1)$$

and

$$\begin{cases} X_j^i(t+1)=f_j^i(X_j^1(t-\tau),\ldots,X_j^N(t-\tau),y_1(t),\ldots,y_M(t)) \\ y_j(t) = g_j(X_j^1(t), \ldots, X_j^N(t)) \end{cases} \quad (2)$$

The main difference of models (1) and (2) is that the communication delay between different BNs is considered in (1) while it is not considered in (2). We can observe that the state evolution of the BNs both for (1) and (2) depends on the initial state sequence $X_j(-\tau), X_j(1-\tau), \ldots, X_j(0), j = 1, 2, \ldots, M$.

Definition 2

The array of BNs in (1) and (2) is said to be synchronized if for any initial states $X_j(-\tau), \ldots, X_j(0) \in \{1,0\}^N, j = 1, \ldots, M$, there is a positive integer k, such that $t \geq k$ satisfies $X_i(t) = X_j(t)$ for any $1 \leq i, j \leq N$.

In this definition, k depends on the initial state sequence $X_j(-\tau), \ldots, X_j(0) \in \{1,0\}^N, j = 1, \ldots, M$. Nevertheless, since the $\{1,0\}^N$ is a finite set, we can by all means choose a k big enough, which is independent of the initial state sequence.

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II. Main Result

We have

$$X_j(t+1) = (F_j^1 \ltimes X_j(t-\tau) \ltimes y(t-\tau)) \ltimes \otimes \ltimes (F_j^N \ltimes X_j(t-\tau) \ltimes y(t-\tau)).$$

Thus, denote $F_j^1\{\ltimes_{i=2}^N [(I_{2^{M+N}} \otimes F_j^i) \ltimes \Phi_{M+N}]\}$ by F_j, we can obtain

$$X_j(t+1) = F_j \ltimes X_j(t-\tau) \ltimes y(t-\tau),\ j = 1, \ldots, M.$$

Similarly, if a 2×2^N matrix G_j is the structure matrix of g_j, then letting $G = \otimes_{j=1}^M G_j$ gives that $y(t) = G \ltimes_{j=1}^M X_j(t)$.

That is to say, we have obtained the following equivalent algebraic representations of BNs (1):

$$\begin{cases} X_j(t+1) = F_j X_j(t-\tau) y(t-\tau) \\ y(t) = G \ltimes_{j=1}^M X_j(t) \end{cases} \quad (3)$$

where F_j is a $2^N \times 2^{MN}$ matrix and G is a $2^M \times 2^{MN}$ matrix.

Lemma 3: Let $W = W_{[2^M, 2^N]} \ltimes \{\ltimes_{i=2}^M [(I_{2^M} \otimes W_{[2^M, 2^{iN}]})\Phi_M]\}$ and $\Xi = (\otimes_{j=1}^M F_j) \ltimes W \ltimes G \ltimes \Phi_{MN}$. Then, we have

$$\ltimes_{j=1}^M X_j(t+1) = \Xi[\ltimes_{j=1}^M X_j(t-\tau)] \quad (4)$$

and

$$\ltimes_{j=1}^M X_j(t) = \Xi^{p+1}[\ltimes_{j=1}^M X_j(q-1-\tau)] \quad (5)$$

where $p \geq 0$ and $1 \leq q \leq \tau + 1$ are the unique integers satisfying $t = p(\tau+1) + q$.

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II. Main Result

Theorem 1

Let (3) be the algebraic representations of the array of delayed BNs (1). Then, synchronization occurs if and only if there exists a positive integer k satisfying $1 \leq k+1 \leq k_0$ such that

$$\mathrm{Col}(\Xi^{k+1}) \subseteq \left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N - 1}, i = 1, 2, \ldots, 2^N\right\} \quad (6)$$

where $k_0 = \min\{i : i \geq 1, \Xi^i = \Xi^j \text{ for some } j > i\}$.

Proof: *(Necessity)* If the array of BNs is synchronized, then we can choose $m \geq 1$ such that $X_i(m) = X_j(m)$ for all $X_j(-\tau), \ldots, X_j(0), j = 1, \ldots, M$ and $1 \leq i, j \leq M$. Moreover, there must exist two unique integers m_1 and m_2 such that $m = m_1(\tau+1) + m_2$. Let $1 \leq r \leq 2^N$ satisfies $X_j(m) = \delta_{2^N}^r$ for each $1 \leq j \leq M$. Then, it follows from Lemma 3 that

$$\Xi^{m_1+1}[\ltimes_{j=1}^{M} X_j(m_2-1-\tau)] = \delta_{2^N}^r \ltimes \delta_{2^N}^r \ltimes \cdots \ltimes \delta_{2^N}^r = \delta_{2^{NM}}^{\lambda_r}$$

where $\lambda_r = 1 + (r-1)(2^{MN}-1)/2^N - 1$. Since $X_1(m_2-1-\tau), X_2(m_2-1-\tau), \ldots, X_M(m_2-1-\tau)$ are arbitrarily given, m_1 satisfies the condition.

Now, let k be the smallest positive integer satisfying the above property (6). To prove the property that $1 \leq k+1 \leq k_0$, we suppose to get a contradiction, that $k+1 > k_0$. Let $s_0 = \min\{i \geq 0 : \Xi^{k_0+i+1} = \Xi^{k_0}\}$. Then, there exists $k_0 \leq l \leq k_0+s_0$ such that $\Xi^l = \Xi^{k+1}$. Since $\mathrm{Col}(\Xi^{k_0}) = \mathrm{Col}(\Xi^{k_0+s_0+1}) \subseteq \mathrm{Col}(\Xi^l) \subseteq \mathrm{Col}(\Xi^{k_0})$, it follows that $\mathrm{Col}(\Xi^{k_0}) = \mathrm{Col}(\Xi^l) = \mathrm{Col}(\Xi^{k+1})$. This contradicts the minimality of k. Therefore, one can conclude that $1 \leq k+1 \leq k_0$.

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II. Main Result

Theorem 1

Let (3) be the algebraic representations of the array of delayed BNs (1). Then, synchronization occurs if and only if there exists a positive integer k satisfying $1 \le k+1 \le k_0$ such that

$$\mathrm{Col}(\Xi^{k+1}) \subseteq \left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i = 1, 2, \ldots, 2^N\right\} \quad (6)$$

where $k_0 = \min\{i : i \ge 1, \Xi^i = \Xi^j \text{ for some } j > i\}$.

Sufficiency: Suppose there exists a positive integer k_1 such that $\mathrm{Col}(\Xi^{k_1+1}) \subseteq \{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + (i-1)(2^{MN}-1)/2^N - 1, i = 1, 2, \ldots, 2^N\}$. Let $k = k_1(\tau+1)+1$. Thus, for every $t \ge k$, there are two unique integers $p \ge 0$ and $1 \le q \le \tau + 1$ satisfying $t = p(\tau+1)+q$. Then, we have $p \ge k_1$, furthermore

$$\mathrm{Col}(\Xi^{p+1}) \subseteq \mathrm{Col}(\Xi^{k_1+1}) \subseteq \left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i = 1, 2, \ldots, 2^N\right\}.$$

It follows from (4) that for every $X_1(q-\tau), \ldots, X_M(q-\tau)$, $0 \le q \le \tau$ and every $t \ge k$, there is $1 \le r' \le 2^N$ such that

$$\ltimes_{j=1}^{M} X_j(t) = \delta_{2^{MN}}^{\lambda_{r'}} = \delta_{2^N}^{\lambda_{r'}} \ltimes \cdots \ltimes \delta_{2^N}^{\lambda_{r'}}.$$

Then, we have $X_j(t) = \delta_{2^N}^{\lambda_{r'}}$ for every $1 \le j \le M$. Hence, the sufficiency is proved. ■

Theorem 1 is also applicable to an array of delay-free BNs by letting $\tau = 0$ in (1).

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II. Main Result

We now consider another type of array of coupled BNs (2), where the outputs $y_j(t)$ do not have time delay. Using the STP, we can get the algebraic representations of BNs (2)

$$\begin{cases} X_j(t+1) = F_j X_j(t-\tau) y(t) \\ y(t) = G \ltimes_{j=1}^{M} X_j(t) \end{cases} \tag{7}$$

where F_j and G can be similarly defined with that of BNs (1).

Lemma 4

The relationship between the state $X_j(t)$ at time t and initial states can be presented as follows.

1) Let $W = W_{[2^M, 2^N]} \ltimes \{\ltimes_{i=2}^{M}[(I_{2^M} \otimes W_{[2^M, 2^{iN}]})\Phi_M]\}$ and $\Xi' = (\otimes_{j=1}^{M} F_j) \ltimes W \ltimes G$. Then

$$\ltimes_{j=1}^{M} X_j(t+1) = \Xi' (\ltimes_{j=1}^{M} X_j(t))(\ltimes_{j=1}^{M} X_j(t-\tau)). \tag{8}$$

---

2) Let $A(t) = (\ltimes_{j=1}^{M} X_j(t)) \ltimes (\ltimes_{j=1}^{M} X_j(t-1)) \ltimes \cdots \ltimes (\ltimes_{j=1}^{M} X_j(t-\tau))$ and

$$\Theta = \Xi' (I_{2^{MN}} \otimes Ed^{MN\tau}) W_{[2^{MN\tau}, 2^{MN(\tau+1)}]} \Phi_{MN\tau}.$$

Then, $A(t+1) = \Theta A(t)$ and

$$\ltimes_{j=1}^{M} X_j(t) = Ed^{MN(\tau+1)} W_{[2^{MN\tau}, 2^{MN}]} \Theta^t (\ltimes_{j=1}^{M} X_j(0)) \ltimes \cdots \ltimes (\ltimes_{j=1}^{M} X_j(1-\tau)) \ltimes (\ltimes_{j=1}^{M} X_j(-\tau))$$

where $Ed^{MN\tau} = (\mathbf{1}_{2^{MN(\tau-1)}} \otimes I_{2^{MN}})$.

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II. Main Result

Theorem 2

Let (7) be the algebraic representations of coupled BNs (2). Then, synchronization occurs if and only if there exists a positive integer k, 1 ≤ k ≤ k0, such that

$$\mathrm{Col}(Ed^{MN(\tau+1)}W_{[2^{MN\tau},2^{MN}]}\Theta^k) \subseteq \left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i = 1, 2, \ldots, 2^N\right\} \quad (9)$$

where $k_0' = \min\{i : i \geq 1, Ed^{MN(\tau+1)}W_{[2^{MN\tau},2^{MN}]}\Theta^i = Ed^{MN(\tau+1)}W_{[2^{MN\tau},2^{MN}]}\Theta^j, j > i\}$.

Proof (Necessity): Suppose that the array of BNs (2) is synchronized, then there is a positive integer k such that for any initial states $X_j(-\tau), \ldots, X_j(0)$, $j = 1, \ldots, M$, we have $X_j(k) = X_i(k)$ for all $1 \leq i, j \leq M$. Let $1 \leq r \leq 2^N$ be such that $X_j(k) = \delta_{2^N}^r$ for each $1 \leq j \leq M$. By Lemma 4, we have

$$\begin{aligned} &Ed^{MN(\tau+1)}W_{[2^{MN\tau},2^{MN}]}\Theta^k(\ltimes_{j=1}^M X_j(0))\cdots(\ltimes_{j=1}^M X_j(-\tau)) \\ &= \delta_{2^N}^r \ltimes \delta_{2^N}^r \ltimes \cdots \ltimes \delta_{2^N}^r \\ &= \delta_{2^{MN}}^{\lambda_r} \end{aligned}$$

where $\lambda_r = 1 + (r-1)(2^{MN}-1)/2^N - 1, 1 \leq r \leq 2^N$. Since $(\ltimes_{j=1}^M X_j(0)), \ldots, (\ltimes_{j=1}^M X_j(-\tau))$ are arbitrarily given, one has

$$\mathrm{Col}(Ed^{MN(\tau+1)}W_{[2^{MN\tau},2^{MN}]}\Theta^k) \subseteq \left\{\delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i = 1, 2, \ldots, 2^N\right\}.$$

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II. Main Result

Theorem 2

Let (7) be the algebraic representations of coupled BNs (2). Then, synchronization occurs if and only if there exists a positive integer k, $1 \le k \le k0$, such that

$$\mathrm{Col}(Ed^{MN(\tau+1)} W_{[2^{MN\tau}, 2^{MN}]} \Theta^{k}) \subseteq \left\{ \delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i = 1, 2, \ldots, 2^N \right\} \quad (9)$$

where $k_0' = \min\{i : i \ge 1, Ed^{MN(\tau+1)} W_{[2^{MN\tau}, 2^{MN}]} \Theta^{i} = Ed^{MN(\tau+1)} W_{[2^{MN\tau}, 2^{MN}]} \Theta^{j}, j > i\}$.

Sufficiency: Suppose that there exists a positive integer k satisfying Property (9). Since

$$\mathrm{Col}(Ed^{MN(\tau+1)} W_{[2^{MN\tau}, 2^{MN}]} \Theta^{t}) \subseteq \left\{ \delta_{2^{MN}}^{\lambda_i} : \lambda_i = 1 + \frac{(i-1)(2^{MN}-1)}{2^N-1}, i = 1, 2, \ldots, 2^N \right\}$$

for $t \ge k$, it follows from Lemma 4 that for every $X_j(-\tau), \ldots, X_j(0)$, $j = 1, \ldots, M$ and every $t \ge k$, there exists $1 \le r' \le 2^N$ such that $\ltimes_{j=1}^{M}(t) = \delta_{2^{MN}}^{\lambda_{r'}} = \delta_{2^N}^{r'} \ltimes \cdots \ltimes \delta_{2^N}^{r'}$, where $\lambda_{r'} = 1 + (r'-1)(2^{MN}-1)/2^N - 1$. Hence, we have $X_j(t) = \delta_{2^N}^{r'}$ for each $1 \le j \le M$. The proof is completed here.

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III. Example

Let $\tau = 1$, $f_1 = (f_1^1, f_1^2, f_1^3) = (\neg x_1^3 \wedge y_2, x_1^1, x_1^2)$, $f_2 = (f_2^1, f_2^2, f_2^3) = (\neg x_2^2 \wedge y_1, x_2^1, x_2^2)$, and $g_1 = \neg x_1^3$, $g_2 = \neg x_2^3$ The algebraic representation of this BN is then expressed as follows:

$$\begin{cases} x_1(t+1) = F x_1(t-1) y_2(t-1) \\ x_2(t+1) = F x_2(t-1) y_1(t-1) \\ y_1(t) = G x_1(t) \\ y_2(t) = G x_2(t) \end{cases} \quad (10)$$

where

$$\begin{cases} F = \delta_8[5, 5, 1, 5, 6, 6, 2, 6, 7, 7, 3, 7, 8, 8, 4, 8] \\ G = \delta_8[2, 1, 2, 1, 2, 1, 2, 1]. \end{cases}$$

Hence, we have

$$\Xi = \delta_{64}[37, 37, 38, 38, 39, 39, 40, 40, 37, 1, 38, 2, 39, 3, 40, 4, 45, 45, 46, 46, 47, 47, 48, 48, 45, 9, 46, 10, 47, 11, 48, 12, 53, 53, 54, 54, 55, 55, 56, 56, 53, 17, 54, 18, 55, 19, 56, 20, 61, 61, 62, 62, 63, 63, 64, 64, 61, 25, 62, 26, 63, 27, 64, 28].$$

Direct computation gives that $\Xi^3 = \Xi^9$, then we have $k_0 = 3$. Moreover, we can obtain that $\Xi^3 \subseteq \{\delta_{64}^i | i = 1,10,19,28,37,46,55,64\}$.Hence, the coupled BNs with time delay can be synchronized using Theorem 1. Fig. 2 shows the total synchronization error $E(t)$.

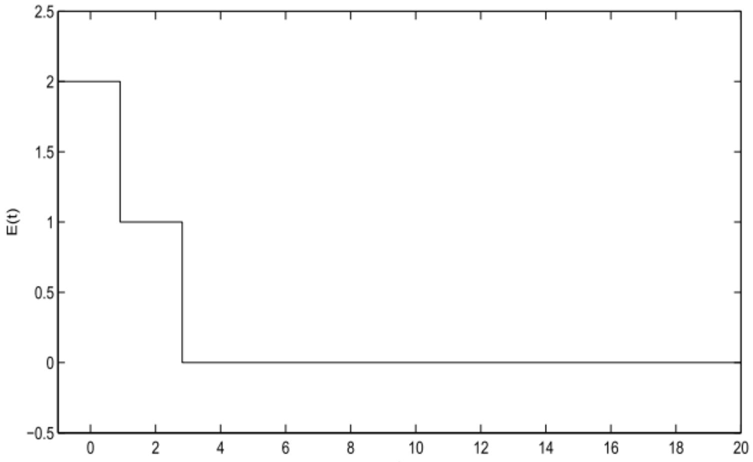

Fig. 2. Total synchronization error of the coupled BNs (10) with initial state sequences $x_1(-1) = (1, 1, 1)$, $x_2(-1) = (0, 0, 0)$, $x_1(0) = (0, 0, 0)$, and $x_2(0) = (1, 1, 1)$.

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End

Asymptotical Stability of PBN With State Delays

- A bridge between asymptotical stability of PBN
- A solution to its induced equations
- Several novel stability criteria
- Universal procedure is summarized

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Markov chain

Consider a Markov chain associated with a state space $[1, n]$ and a state transition matrix $\mathbf{P}$, here and elsewhere $\mathbf{P}^\top \in \mathscr{P}^{n\times n}$. Vector $\mathbf{r} \in \mathscr{P}^{1\times n}$ is called a *stationary distribution* if it satisfies $\mathbf{r} = \mathbf{rP}$.

For every initial state i and destination state j, the *first arrival time* from i to j is defined as $\mathrm{T}_{i\to j} := \mathrm{argmin}_{k\geq 1}\{k \in \mathscr{N} : [\mathbf{P}^k]_{ij} > 0\}$, where $[A]_{ij} := (\delta_n^i)^\top A \delta_n^j$ for matrix $A \in \mathscr{R}^{n\times n}$. Particularly, if such an integer k does not exist, let $\mathrm{T}_{i\to j} := \infty$. The *first arrival probability* from i to j at the k-time step is denoted by $f_{i\to j}^k := P\{[\mathbf{P}^k]_{ij} > 0, [\mathbf{P}^t]_{ij} = 0, t \in [1, k-1]\}$, where $P\{\mathbf{A}\}$ represents the probability of event $\mathbf{A}$ occurring. Following it, the *first arrival probability* from i to j is computed as $f_{i\to j} := \sum_{k=1}^{\infty} f_{i\to j}^k$. Let $d(i)$ be the maximum common divisor of the integer set $\{k \in \mathscr{N} : [\mathbf{P}^k]_{ii} > 0\}$; it is called the *period* of state i henceforth. If $d(i) > 1$, state i is *periodic*; otherwise (i.e., $d(i) = 1$), it is *aperiodic*.

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In accordance with the first arrival time and the first arrival probability, the state space $[1, n]$ can be classified into different categories: for each state $i \in [1, n]$, it is called a *recurrent state*, if $f_{i \to i} = 1$; otherwise (i.e., $f_{i \to i} < 1$), it is called a *transition state*. More precisely, as for a recurrent state i, it is said to be *positive*, if $\mu_i := \sum_{k=1}^{\infty} k f_{i \to i}^{k} < +\infty$; otherwise (i.e., $\mu_i = +\infty$), it is called *null*.

For each state pair $i, j \in [1, n]$, if there exists a positive integer k such that $\left[\mathbf{P}^k\right]_{ij} > 0$, then j is said to be *reachable* from i, denoted by $i \to j$; states i and j are called *connected*, denoted by $i \leftrightarrow j$, if $i \to j$ and $j \to i$. A Markov chain with each pair of states being connected is called *irreducible*.

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Property 1 (see [1])

1) For each state pair $i, j \in [1, n]$, it holds that $\lim_{k\to\infty} \left[\mathbf{P}^k\right]_{ij} =$

$$\begin{cases} 0, & j \text{ is a transition state or null recurrent state} \\ \frac{f_{i\to j}}{\mu_j}, & j \text{ is a positive recurrent state with } d(j) = 1. \end{cases}$$

Besides, $\lim_{k\to\infty} \left[\mathbf{P}^k\right]_{jj}$ does not exist for any positive recurrent state j with $d(j) > 1$.

2) Finite-state Markov chains does not have null recurrent state.
3) All recurrent states make up of a closed set $\mathscr{C}_0$.
4) An irreducible aperiodic positive recurrent Markov chain has a unique stationary distribution $\mathbf{r}$, where $r_i > 0$ for each $i \in [1, n]$. More precisely, stationary distribution $\mathbf{r}$ is exactly the limit distribution of this Markov chain.
5) For two connected states i and j (i.e., $i \leftrightarrow j$), they process the same periodic and recurrent properties.

By the connection relationships of distinct states, one can split set $\mathscr{C}_0$ into a series of *basic recurrent closed sets* $\mathscr{C}_i, i = 1, 2, \ldots$, satisfying all conditions:

1) $\bigcup_i \mathscr{C}_i = \mathscr{C}_0$;

2) $\mathscr{C}_j \cap \mathscr{C}_k = \emptyset, j \neq k$, that is, arbitrary state in $\mathscr{C}_j$ cannot be mutually connected with any state in $\mathscr{C}_k$;

3) different states in the same set $\mathscr{C}_i$ are mutually connected. Subsequently, the state space $[1, n]$ can be decomposed as $[1, n] \coloneqq \mathscr{H} \cup \mathscr{C}_0$ with $\mathscr{C}_0 \coloneqq \mathscr{C}_1 \cup \mathscr{C}_2 \cup \cdots$, where the set $\mathscr{H}$ consists of all transition states.

[1] S. M. Ross, Stochastic Processes. New York, NY, USA: Wiley, 1996.
[2] Zhu S, et al. Asymptotical Stability of Probabilistic Boolean Networks With State Delays[J].2020

I. Problem Formulation

An n nodes' PBN with bounded and coincident state delays reads

$$x_i(t+1) = f_i\left(x_1(t-\tau(t)), \ldots, x_n(t-\tau(t))\right), \ i \in [1, n] \qquad (1)$$

where $x_i(t) \in \mathscr{D}$, $i \in [1, n]$, are also called states, which can be easily distinguished from those of the Markov chain from the context. State delay $\tau(t)$ is randomly chosen from $[0, \gamma]$ with respect to the probability distribution $[\rho_0, \rho_1, \ldots, \rho_\gamma]$.

By bijection ϱ, x_i can be expressed as a delta form in Δ_2. Let $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{2^n}$; denote by $\mathbb{E}\{x(t)\}$ (respectively, $\mathbb{E}\{z(t)\}$) the overall mathematical expectation of the state variable $x(t)$ (respectively, $z(t)$); from Lemma 1 and the property of the swap matrix, one can convert the PBN (1) into the following algebraic representation:

$$\mathbb{E}\{x(t+1)\} = \mathbf{F} \ltimes \mathbb{E}\{x(t-\tau(t))\} \qquad (2)$$

where $\mathbf{F} \in \mathscr{P}^{2^n \times 2^n}$ is called the transition matrix of the PBN (1).

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I. Problem Formulation

Definition 2: For a given $x_d \in \Delta_{2^n}$, the PBN (2) is said to be globally asymptotically stable at x_d, if

$$\lim_{t\to\infty} \left\| \mathbb{E}\left\{x\left(t; x(0), x(-1), \ldots, x(-\gamma)\right)\right\} - x_d \right\|_2 = 0 \qquad (3)$$

holds for all $x(0), x(-1), \ldots, x(-\gamma) \in \Delta_{2^n}$, where $\|\cdot\|_2$ is the Euclidean norm, and the state trajectory of the PBN (2) with initial states $x(-k; x(0), x(-1), \ldots, x(-\gamma)) = x(-k)$, $k \in [0, \gamma]$, is recorded as $x\left(t; x(0), x(-1), \ldots, x(-\gamma)\right)$.

Remark 1: By viewing x_d as a one-point distribution, an alternative description of formula (3) is

$$\lim_{t\to\infty} P\left\{x(t; x(0), x(-1), \ldots, x(-\gamma)) = x_d\right\} = 1. \qquad (4)$$

In a more generalized case of $x_d \in \mathscr{P}^{2^n \times 1}$, formula (3) still works. As for (4), a little modification is requisite, as

$$\lim_{t\to\infty} P\left\{x\left(t; x(0), x(-1), \ldots, x(-\gamma)\right) = \delta_{2^n}^j\right\} = [x_d]_j$$

holds for all $j \in [1, 2^n]$ and $x(0), x(-1), \ldots, x(-\gamma) \in \Delta_{2^n}$, where $[\mathbf{v}]_j := (\delta_w^j)^\top \mathbf{v}$ for $w \times 1$ vector $\mathbf{v}$.

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II. The influence of state delays on asymptotical stability

In what follows, to facilitate the analysis, a novel argument system is constructed to eliminate the appearance of state delays in the form. It begins with the expanding of formula (2) as

$$
\begin{aligned}
&\mathbb{E}\{x(t+1)\}\\
&= \rho_0 \mathbf{F}\mathbb{E}\{x(t)\} + \rho_1 \mathbf{F}\mathbb{E}\{x(t-1)\} + \cdots + \rho_\gamma \mathbf{F}\mathbb{E}\{x(t-\gamma)\}\\
&= \sum_{i=0}^{\gamma} \left(\rho_i \mathbf{F}(\mathrm{E}_{[2^n]}\mathrm{W}_{[2^n]})^{\gamma-i}\mathrm{E}^i_{[2^n]}\mathbb{E}\{\ltimes_{j=0}^{\gamma} x(t-j)\}\right)\\
&= \left(\sum_{i=0}^{\gamma} \rho_i \mathbf{F}(\mathrm{E}_{[2^n]}\mathrm{W}_{[2^n]})^{\gamma-i}\mathrm{E}^i_{[2^n]}\right)\mathbb{E}\left\{\ltimes_{j=0}^{\gamma} x(t-j)\right\}\\
&:= \mathbf{M}\mathbb{E}\left\{\ltimes_{j=0}^{\gamma} x(t-j)\right\}
\end{aligned}
$$

where $\mathbf{M} := \sum_{i=0}^{\gamma} \rho_i \mathbf{F}(\mathrm{E}_{[2^n]}\mathrm{W}_{[2^n]})^{\gamma-i}\mathrm{E}^i_{[2^n]} \in \mathscr{P}^{2^n \times 2^{n(\gamma+1)}}$.
Afterward, setting $z(t) = \ltimes_{j=0}^{\gamma} x(t-j)$, one obtains that

$$
\begin{aligned}
&\mathbb{E}\{z(t+1)\}\\
&= \mathbb{E}\{\ltimes_{j=0}^{\gamma} x(t-j+1)\}\\
&= \mathbb{E}\{x(t+1)\left(\mathrm{E}_{[2^n]}\mathrm{W}_{[2^{n\gamma},2^n]}\right) \ltimes_{j=0}^{\gamma} x(t-j)\}\\
&= \mathbf{M}\mathbb{E}\{z(t)\left(\mathrm{E}_{[2^n]}\mathrm{W}_{[2^{n\gamma},2^n]}\right) z(t)\}\\
&= \mathbf{M}\left(I_{2^{n(\gamma+1)}} \otimes \left(\mathrm{E}_{[2^n]}\mathrm{W}_{[2^{n\gamma},2^n]}\right)\right)\Phi_{2^{n(\gamma+1)}}\mathbb{E}\{z(t)\}\\
&:= \mathbf{\Gamma}\mathbb{E}\{z(t)\} \qquad (5)
\end{aligned}
$$

where $\mathbf{\Gamma} := \mathbf{M}\left(I_{2^{n(\gamma+1)}} \otimes \left(\mathrm{E}_{[2^n]}\mathrm{W}_{[2^{n\gamma},2^n]}\right)\right)\Phi_{2^{n(\gamma+1)}} \in \mathscr{P}^{2^{n(\gamma+1)} \times 2^{n(\gamma+1)}}$. It can be regarded as the transition matrix of a higher dimensional PBN (5).

Normal method to deal with time delay

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III. Asymptotical stability of a PBN

Theorem 1: For a given $x_d \in \Delta_{2^n}$, the PBN (2) is said to be globally asymptotically stable at x_d if and only if $\mathbf{b} = [z_d^\top, 1]^\top$ is the unique nonnegative solution to equations

$$\begin{cases} \mathbf{b} = \mathbf{L}\mathbf{b} \\ [\mathbf{b}]_{2^{n(\gamma+1)}+1} = 1 \end{cases} \tag{6}$$

where $\mathbf{L} := \begin{bmatrix} \Gamma & \mathbf{0}_{2^{n(\gamma+1)}} \\ \mathbf{1}^\top_{2^{n(\gamma+1)}} & 0 \end{bmatrix}$ and $z_d = \underbrace{x_d \ltimes \cdots \ltimes x_d}_{\gamma+1}$.

Remark 2: In (6), the coefficient matrix $\mathbf{L}$ can be uniquely determined by the transition matrix Γ of the PBN (5). Accordingly, equations in (6) are termed as the induced equations of the PBN (5) henceforth.

[1] S. M. Ross, Stochastic Processes. New York, NY, USA: Wiley, 1996.
[2] Zhu S, et al. Asymptotical Stability of Probabilistic Boolean Networks With State Delays[J].2020

III. Asymptotical stability of a PBN

Proposition 1: For a given $z_d \in \mathscr{P}^{2^{n(\gamma+1)} \times 1}$, the PBN (5) is said to be globally asymptotically stable at z_d if and only if $\mathbf{b} = [z_d^\top, 1]^\top$ is the unique nonnegative solution to induced equations (6) and $d(i) = 1$ for all $i \in G$, where $G = \{j \in [1, 2^{n(\gamma+1)}] : [z_d]_j > 0\}$.

Proof: [Sufficiency] A little deduction can verify that the PBN (5) has a unique stationary distribution z_d. Following this observation, one can confirm that the PBN (5) has a unique basic recurrent closed set $\mathscr{C}_1$. Without loss of generality, two basic recurrent closed sets $\mathscr{C}_1$ and $\mathscr{C}_2$ are assumed. That is, one can find a permutation matrix $M \in \mathscr{L}^{2^{n(\gamma+1)} \times 2^{n(\gamma+1)}}$, which satisfies $M^\top M = I_{2^{n(\gamma+1)}}$, such that

$$M^\top \mathbf{\Gamma} M = \begin{bmatrix} \mathbf{\Gamma}_1 & * & \mathbf{R}_1 \\ * & \mathbf{\Gamma}_2 & \mathbf{R}_2 \\ * & * & \mathbf{Q} \end{bmatrix}$$

where "$*$" represents the appropriate dimensional submatrix block with all entries being 0. Since the submatrix block $\mathbf{\Gamma}_1$ is still a probability matrix, $\mathbf{\Gamma}_1^\top$ can act as a state transition matrix of a Markov chain, which has at least one stationary distribution, denoted by $\mathbf{r}^1$. Extend $(\mathbf{r}^1)^\top$ into $[\mathbf{r}^1, *]^\top \in \mathscr{P}^{2^{n(\gamma+1)} \times 1}$ and make $\bar{\mathbf{r}}^1 = M[\mathbf{r}^1, *]^\top$; it is obviously a stationary distribution of the PBN (5). Analogously, the submatrix block $\mathbf{\Gamma}_2$ also corresponds to at least a stationary distribution $\mathbf{r}^2$. Hence, vector $\bar{\mathbf{r}}^2 = M[*, \mathbf{r}^2, *]^\top$ is also a stationary distribution of the PBN (5). From the solution uniqueness of induced equations (6), one has $[\mathbf{r}^1, *]^\top = [*, \mathbf{r}^2, *]^\top$. This is a contradiction, because $\mathbf{0}_{2^n}$ is not a solution to (6). Thus, there is only one basic recurrent closed set $\mathscr{C}_1$. Besides, as 2) in Property 1 implies, all states in $\mathscr{C}_1$ are positive recurrent states. That is, a permutation matrix $M \in \mathscr{L}^{2^{n(\gamma+1)} \times 2^{n(\gamma+1)}}$ can be found to satisfy

$$M^\top \mathbf{\Gamma} M = \begin{bmatrix} \mathbf{\Gamma}_1 & \mathbf{R}_1 \\ * & \mathbf{Q} \end{bmatrix}. \qquad (7)$$

To proceed, this basic recurrent closed set $\mathscr{C}_1$ is verified to be G. First, one proves that $G \subseteq \mathscr{C}_1$. If there exists an integer $i \in G \backslash \mathscr{C}_1$, state i must be a transition state. By 1) in Property 1, $[z_d]_i = \lim_{k \to +\infty} \mathrm{Row}_i (\mathbf{\Gamma}^k) z_d = 0$, which contradicts with the fact $i \in G$. Afterward, one verifies that $G = \mathscr{C}_1$. According to $d(i) = 1$ for $i \in G$ and $G \subseteq \mathscr{C}_1$, by 5) of Property 1, the Markov chain associated with the state transition matrix $\mathbf{\Gamma}_1^\top$ is an irreducible aperiodic positive recurrent chain. According to 4) in Property 1, it has a unique stationary distribution $\mathbf{r}_1$. Moreover, the vector $\bar{\mathbf{r}}^1 = M[\mathbf{r}^1, *]^\top$ is a stationary distribution of the PBN (5). Assume there is a state $i \in \mathscr{C}_1 \backslash G$; one

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III. Asymptotical stability of a PBN

Proposition 1: For a given $z_d \in \mathscr{P}^{2^{n(\gamma+1)} \times 1}$, the PBN (5) is said to be globally asymptotically stable at z_d if and only if $\mathbf{b} = [z_d^\top, 1]^\top$ is the unique nonnegative solution to induced equations (6) and $d(i) = 1$ for all $i \in G$, where $G = \{j \in [1, 2^{n(\gamma+1)}] : [z_d]_j > 0\}$.

T.B.C.

concludes that $M[\mathbf{r}^1, *]^\top \neq z_d$, because the ith entries are distinct. It contradicts with the solution uniqueness of induced equations (6). As such, the proof of $\mathscr{C}_1 = G$ is complete.

Finally, the PBN (5) is confirmed to be globally asymptotically stable at z_d. Utilizing the decomposition of the state space as $[1, 2^{n(\gamma+1)}] := \mathscr{H} \cup \mathscr{C}_1$, set $\mathscr{H}$ consists of all transition states. By 1) in Property 1, it is claimed that $\lim_{k\to+\infty} [\mathbf{\Gamma}^k]_{ij} = 0$, for each $i \in \mathscr{H}$ and $j \in [1, 2^{n(\gamma+1)}]$. Since all states in $\mathscr{C}_1$ are aperiodic positive recurrent states, it is concluded that $\lim_{k\to+\infty} [\mathbf{\Gamma}^k]_{ij} = [z_d]_i$ for each $i \in \mathscr{C}_1$ and $j \in [1, 2^{n(\gamma+1)}]$, by 4) of Property 1. To sum up, the PBN (5) is globally asymptotically stable at z_d. The proof of sufficiency is complete.

[Necessity] Suppose that the PBN (5) is globally asymptotically stable at z_d; if the number of basic recurrent closed sets is more than one, there must exist a permutation matrix $M \in \mathscr{L}^{2^{n(\gamma+1)} \times 2^{n(\gamma+1)}}$ such that

$$M^\top \mathbf{\Gamma} M = \begin{bmatrix} \mathbf{\Gamma}_1 & * & * & \mathbf{R}_1 \\ * & \mathbf{\Gamma}_2 & * & \mathbf{R}_2 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & * & \mathbf{Q} \end{bmatrix}.$$

Following that, each state in the basic recurrent closed set $\mathscr{C}_1$ will not enter the basic recurrent closed set $\mathscr{C}_2$, and this fact is also true for states in $\mathscr{C}_2$. This is a contradiction, as the PBN (5) is globally stable. Therefore, the state-space decomposition is presented in the form as (7), and the basic recurrent closed set is unique.

Additionally, the Markov chain associated with the state transition matrix $\mathbf{\Gamma}_1^\top$ is an irreducible aperiodic positive recurrent chain. By 4) of Property 1, it has a unique stationary distribution $\mathbf{r}^1$, which acts as its limit distribution.

Extend vector $\mathbf{r}^1$ to $[\mathbf{r}^1, *]^\top$ and make $\tilde{\mathbf{r}}^1 = M[\mathbf{r}^1, *]^\top$; the vector obtained by adding 1 at the bottom of $\tilde{r}^1$ is a feasible solution of (6). Besides, $d(i) = 1$ for each $i \in G$. The existence of such a solution is proved. Furthermore, the stationary distribution $\mathbf{r}$ of the PBN (5) should satisfy $[\mathbf{r}]_i = 0$ for each $i \in \mathscr{H}$; thus, the PBN (5) has a unique stationary distribution, namely $\tilde{\mathbf{r}}^1 = z_d$. The proof of necessity is complete. ■

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III. Asymptotical stability of a PBN

Proposition 1: For a given $z_d \in \mathscr{P}^{2^{n(\gamma+1)} \times 1}$, the PBN (5) is said to be globally asymptotically stable at z_d if and only if $\mathbf{b} = [z_d^\top, 1]^\top$ is the unique nonnegative solution to induced equations (6) and $d(i) = 1$ for all $i \in G$, where $G = \{j \in [1, 2^{n(\gamma+1)}] : [z_d]_j > 0\}$.

Theorem 2: For a given $z_d \in \Delta_{2^{n(\gamma+1)}}$ with $z_d := \delta^r_{2^{n(\gamma+1)}}$, the PBN (5) is said to be globally asymptotically stable at z_d if and only if $[\mathbf{\Gamma}]_{rr} = 1$ and each solution λ to equation $\det(\lambda I_{2^{n(\gamma+1)}-1} - \mathbf{Q}) = 0$ satisfies $[\mathrm{Re}(\lambda)]^2 + [\mathrm{Im}(\lambda)]^2 < 1$, where $\mathbf{Q}$ is the matrix obtained from $\mathbf{\Gamma}$ by deleting the rth row and column.

Proof: The PBN (5) is globally asymptotically stable at z_d if and only if $\delta^r_{2^{n(\gamma+1)}}$ is a fixed point and the Markov chain with the state transition matrix $\mathbf{\Gamma}^\top$ is an absorbing chain. It has been proved in [31] that a Markov chain is absorbing if and only if the spectral radius $\rho(\mathbf{Q}) < 1$. It amounts to say that $[\mathrm{Re}(\lambda)]^2 + [\mathrm{Im}(\lambda)]^2 < 1$ for each solution λ. ■

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IV. Influence of state delays on asymptotical stability

Lemma 2 (see [25]): The PBN $\mathbb{E}\{x(t+1)\} = \mathbf{F} \ltimes \mathbb{E}\{x(t)\}$ is globally $\delta_{2^n}^r$ -stable in distribution, if and only if,

1) $\delta_{2^n}^r$ is a fixed point, that is, $[\mathbf{F}]_{rr} = 1$;
2) there exists an admissible path with length $l_{x_0} \leq 2^n - 1$ from initial state x_0 to target state $\delta_{2^n}^r$, for each $x_0 \in \Delta_{2^n}$.

Theorem 3: For a given $x_d \in \Delta_{2^n}$, the PBN (2) is globally asymptotically stable at x_d if and only if the PBN $\mathbb{E}\{x(t+1)\} = \mathbf{F} \ltimes \mathbb{E}\{x(t)\}$ is globally asymptotically stable at x_d.

Proof: [Necessity] Suppose the PBN $\mathbb{E}\{x(t+1)\} = \mathbf{F} \ltimes \mathbb{E}\{x(t)\}$ is not globally asymptotically stable at x_d, where $x_d = \delta_{2^n}^r$. In accordance with Lemma 2, only two situations are admissible.

1) $\delta_{2^n}^r$ is not a fixed point, that is, $[\mathbf{F}]_{rr} = \varepsilon < 1$.
2) There exists a state in Δ_{2^n} , from which any path cannot reach $\delta_{2^n}^r$.

Discuss case 1). Let t at both sides of $\mathbb{E}\{x(t+1)\} = \mathbf{F}\mathbb{E}\{x(t-\tau(t))\}$ approaches infinity; according to Definition 2, one has that $\delta_{2^n}^r = \mathbf{F}\delta_{2^n}^r = \mathrm{Col}_r(\mathbf{F})$. This is a contradiction, as $[\delta_{2^n}^r]_r = 1$.

[Sufficiency] Since the PBN $\mathbb{E}\{x(t+1)\} = \mathbf{F} \ltimes \mathbb{E}\{x(t)\}$ is supposed to be globally asymptotically stable at x_d, then $\lim_{t\to\infty}\mathbf{F}^t = \mathbf{1}_{2^n}^\top \otimes \delta_{2^n}^r$. For both sides of the PBN (2), let t approaches infinity; it holds that

$$\lim_{t\to+\infty}\mathbb{E}\{x(t+1)\}$$
$$= \sum_{i=0}^{\gamma} p_i \left(\mathbf{1}_{2^n}^\top \otimes \delta_{2^n}^r\right) \left[\frac{1}{2^n}, \frac{1}{2^n}, \ldots, \frac{1}{2^n}\right]^\top = \delta_{2^n}^r$$

where p_i is the probability of the initial state being $x(-i)$ and $\sum_{i=0}^{\gamma} p_i = 1$. Thus, the proof of sufficiency is complete. ■

[25] Guo Y, et al. Stability and set stability in distribution of probabilistic Boolean networks[J].2019.
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1) $\delta_{2^n}^r$ is a fixed point, that is, $[\mathbf{F}]_{rr} = 1$;
2) there exists an admissible path with length $l_{x_0} \leq 2^n - 1$ from initial state x_0 to target state $\delta_{2^n}^r$, for each $x_0 \in \Delta_{2^n}$.

Corollary 1: For a given $x_d \in \Delta_{2^n}$, the PBN (2) is globally asymptotically stable at x_d if and only if $\mathbf{c} = [x_d^\top, 1]^\top$ is the unique nonnegative solution to equations

$$\begin{cases} \mathbf{c} = \mathbf{L}'\mathbf{c} \\ [\mathbf{c}]_{2^n+1} = 1 \end{cases} \quad (9)$$

where $\mathbf{L}' = \begin{bmatrix} \mathbf{F} & \mathbf{0}_{2^n} \\ \mathbf{1}_{2^n}^\top & 0 \end{bmatrix}$.

Theorem 3: For a given $x_d \in \Delta_{2^n}$, the PBN (2) is globally asymptotically stable at x_d if and only if the PBN $\mathbb{E}\{x(t+1)\} = \mathbf{F} \ltimes \mathbb{E}\{x(t)\}$ is globally asymptotically stable at x_d.

[25] Guo Y, et al. Stability and set stability in distribution of probabilistic Boolean networks[J].2019.
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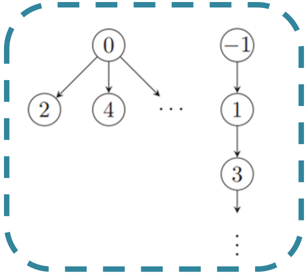
Summary

$$x(t+1) = L_0 u(t-\mu+1) z(t-\mu+1),$$

$$z(t) = \ltimes_{i=t}^{t+\mu-1} x(i) \in \Delta_{2^{\mu n}}$$

$$z(t+1) = L u(t) z(t),$$

Common methods of dimension extension



The constructed forest

Controllability matrices

$$(\mathbf{C}(t,t+k))_{ij} = \begin{cases} 1, & \delta_{2^n}^i \in \mathcal{R}^{(t,t+k)}(\delta_{2^n(\tau+1)}^j) \\ 0, & \delta_{2^n}^i \notin \mathcal{R}^{(t,t+k)}(\delta_{2^n(\tau+1)}^j). \end{cases}$$

$$(\mathbf{C}_t)_{ij} = \begin{cases} 1, & \delta_{2^n}^i \in \mathcal{R}^t(\delta_{2^n(\tau+1)}^j) \\ 0, & \delta_{2^n}^i \notin \mathcal{R}^t(\delta_{2^n(\tau+1)}^j). \end{cases}$$

Completeness and Normal Form of Multi-valued Logical Functions

Main Results:

- Using algebraic form, a method is proposed to construct an adequate set of connectives (ASC) for k-valued logical functions, which can be used to express any k-valued logical functions.
- The disjunctive normal form and conjunctive normal form of k-valued logical functions are presented based on ASC. The ASC is then simplified to a condensed set.
- The normal forms are further extended to mix-valued logical functions

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I. The algebraic form of k-valued logical functions

Definition 2.2. A k-valued logical variable χ takes its values i

$$\mathcal{D}_k := \left\{1, \frac{k-2}{k-1}, \frac{k-3}{k-1}, \ldots, \frac{1}{k-1}, 0\right\}, \quad k \geq 3.$$

We identify each logical value (α_i) with a vector (a_i) as

$$\alpha_i := \frac{k-i}{k-1} \Leftrightarrow a_i = \delta_k^i, \quad i = 1, 2, \ldots, k.$$

Denote by Φ_k the set of unary operators on $\mathcal{D}_k$.

Let $\sigma \in \Phi_k$. Then there exists a unique $\beta_\sigma = [\alpha_{i_1}, \ldots, \alpha_{i_k}]$, such that

$$\sigma(\alpha_i) = \alpha_{i_j}, \quad 1 \leq j \leq k.$$

Moreover, if χ is expressed in vector form as $x \in \Delta_k$, then $\sigma \in \Phi_k$ has matrix form $M_\sigma = \delta_k[i_1, i_2, \ldots, i_k]$.

Now if $\chi \in \mathcal{D}_k$, $\sigma \in \Phi_k$, and assume

$$\xi = \sigma(\chi).$$

Then in vector form we have

$$y = M_\sigma x,$$

where x and y are vector forms of χ and ξ respectively.

Assume $F : \mathcal{D}_k^n \to \mathcal{D}_k$ is an n-variable k-valued logical function. Assume its algebraic form is

$$F(x_1, \ldots, x_n) := M_F \ltimes_{i=1}^n x_i, \tag{24}$$

where $M_F \in \mathcal{L}_{k \times k^n}$ is the structure matrix of F. Then we split M_F into k^{n-1} blocks as

$$M_F := [M_1, M_2, \ldots, M_{k^{n-1}}], \tag{25}$$

where $M_j \in \mathcal{L}_{k \times k}$, $j = 1, \ldots, k^{n-1}$.

Then we define a set of unary operators as

$$\phi_j \in \Phi_k, \quad j = 1, \ldots, k^{n-1}, \tag{26}$$

which have M_j as their structure matrices respectively.

$\{\neg, \vee, \wedge\}$

➢ **Adequate Set of Connectives (ASC)** : a set of logic generators such that any logical function can be expressed as a compounded function of this set of generators.

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II. Normal Form and ASC

Similarly to the Boolean case, we have the following result.

Theorem 3.1: Every k-valued logical function has its disjunctive normal form and its conjunctive normal form.

Proof: We give a constructive proof for them. Assume a k-valued logical function $F(\chi_1, \cdots, \chi_k) : \mathcal{D}_k^n \to \mathcal{D}_k$ is given. Moreover, assume that the structure matrix of F is

$$M_F = [N_1, N_2, \cdots, N_k],$$

where $N_i \in \mathcal{L}_{k \times k^{n-1}}$, $i = 1, \cdots, k$. Similar to Boolean case, it is easy to prove that

$$\begin{aligned} F(\chi_1, \chi_2, \ldots, \chi_k) = & \left[\triangleright_k^1(\chi_1) \wedge F_1(\chi_2, \ldots, \chi_k)\right] \\ & \vee \left[\triangleright_k^2(\chi_1) \wedge F_2(\chi_2, \ldots, \chi_k)\right] \\ & \vee \ldots \\ & \vee \left[\triangleright_k^k(\chi_1) \wedge F_k(\chi_2, \ldots, \chi_k)\right], \end{aligned}$$

where F_i has N_i as its structure matrix, $i = 1, \ldots, k$.

By applying this procedure to each F_i and their sub-functions and using Eq. (26) at the last step, we have

- Disjunctive normal form:

$$\begin{aligned} F(\chi_1, \ldots, \chi_n) = & \bigvee_{i_1=1}^k \bigvee_{i_2=1}^k \cdots \bigvee_{i_{n-1}=1}^k \Big[\triangleright_k^{i_1}(\chi_1) \\ & \bigwedge \triangleright_k^{i_2}(\chi_2) \bigwedge \ldots \bigwedge \triangleright_k^{i_{n-1}}(x_{n-1}) \bigwedge \phi^{i_1, i_2, \ldots, i_k}(\chi_n) \Big] \end{aligned} \quad (28)$$

where $\phi^{i_1, i_2, \ldots, i_{n-1}} = \phi_j$ with

$$j = (i_1 - 1)k^{n-2} + (i_2 - 1)k^{n-3} + \cdots + (i_{n-2} - 1)k + i_{n-1}.$$

- Conjunctive normal form:

Assume $\neg F(\chi_1, \cdots, \chi_n)$ has disjunctive normal form as (28). Then using De Morgan formula, we can have

$$\begin{aligned} F(\chi_1, \cdots, \chi_n) = & \bigwedge_{i_1=1}^k \bigwedge_{i_2=1}^k \cdots \bigwedge_{i_{n-1}=1}^k \\ & \Big[\triangleleft_k^{i_1}(\chi_1) \bigvee \triangleleft_k^{i_2}(\chi_2) \bigvee \cdots \bigvee \triangleleft_k^{i_{n-1}}(\chi_{n-1}) \\ & \bigvee \phi^{k+1-i_1, k+1-i_2, \cdots, k+1-i_k}(\chi_n) \Big]. \end{aligned} \quad (29)$$

Note that in (28) and (29), $\vee$, $\wedge$, and $\neg$ are brief forms for $\vee^{(k)}$, $\wedge^{(k)}$, and $\neg^{(k)}$ respectively.

Corollary 3.2: For k-valued logic

$$A_{cd}^k := \{\wedge^{(k)}, \vee^{(k)}, \Phi_k\} \quad (30)$$

is an ASC.

II. Normal Form and ASC

✓ **In the following we consider how to reduce the size of this ASC.**

Define

$$\Psi_k^n := \{\sigma \in \Phi_k \mid \det(M_\sigma) \neq 0\},$$
$$\Psi_k^s := \{\sigma \in \Phi_k \mid \det(M_\sigma) = 0\}.$$

Then

$$\Phi_k = \Psi_k^n \cup \Psi_k^s.$$

Denote by

$$M_k^n := \{M_\sigma \mid \sigma \in \Psi_k^n\},$$
$$M_k^s := \{M_\sigma \mid \sigma \in \Phi_k^s\}.$$

We define a mapping $\pi : \Phi_k \to M_k$ as

$$\pi : \sigma \mapsto M_\sigma.$$

By restricting it on Ψ_k^n and Ψ_k^s we have $\pi : \Psi_k^n \to M_k^n$ and $\pi : \Psi_k^s \to M_k^s$ respectively.

Then the following relations are obvious.

Proposition 3.3: (i) $\pi : \Psi_k^n \to M_k^n$ is a group isomorphism.

(ii) $\pi : \Psi_k^s \to M_k^s$ is a semigroup isomorphism.

Because of these isomorphisms, instead of Ψ_k^n and Ψ_k^s, we can investigate M_k^n and M_k^s respectively.

Next, we consider Ψ_k^s. We first define an equivalence on Ψ_k^s as follows:

Definition 3.5.

(i) Two unary operators $\sigma_1,\ \sigma_2 \in \Psi_k^s$ are said to be equivalent, and this is denoted by $\sigma_1 \sim \sigma_2$, if there exist $\mu_1,\ \mu_2 \in \Psi_k^n$, such that

$$\sigma_1 \circ \mu_1 = \mu_2 \circ \sigma_2. \quad (31)$$

(ii) Two matrices $M_1,\ M_2 \in M_k^s$ are said to be equivalent, and this is denoted by $M_1 \sim M_2$, if there exist $P_1,\ P_2 \in M_k^n$, such that

$$M_1 P_1 = P_2 M_2. \quad (32)$$

The following proposition is obvious.

Proposition 3.6. *Assume $\sigma,\ \mu \in \Psi_k^s$. Then*

$$\sigma \sim \mu \Leftrightarrow M_\sigma \sim M_\mu.$$

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II. Normal Form and ASC

✓ If two operators are equivalent, then one can be generated from the other one. Then we **only need to choose one representative operator from each equivalence class** as an element of the generator set, denoted by G_k^s.

According to Proposition 3.6, we need only to consider the equivalence class on M_k^s. Two rows are said to be equivalent (denoted by $\sim$), if they have same numbers of "1". For example, $(1,1,0,0) \sim (0,1,0,1)$.

Let $M_1,\ M_2 \in M_k^s$. M_1 and M_2 are said to be equivalent, denoted by $M_1 \sim M_2$, if they have one-one corresponding equivalent rows. For example, we have

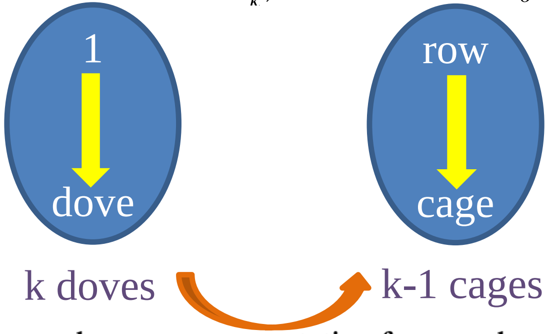
$$M_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad (33)$$

because

$$Row_1(M_1) \sim Row_3(M_2),$$
$$Row_2(M_1) \sim Row_1(M_2),$$
$$Row_3(M_1) \sim Row_2(M_2).$$

Dove-cage problem:

Consider "1" as a dove, "row" as a cage, the number of equivalence classes, denoted by $n(k)$, is equivalent to the following classical dove-cage problem: put k doves into $k-1$ cages. This is because $\sigma \in \Psi_k^s$, which means M_σ is singular.



Then we can choose a representative from each equivalence class to form a set of generators, denoted by G_k^s. Hence, we have $|G_k^s| = n(k)$.

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III. The normal forms of mix-valued logical functions

Definition 4.1: Assume $\chi_i \in \mathcal{D}_{k_i}$, $i = 1, \cdots, n$, a mapping $F : \prod_{i=1}^{n} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$, denoted by $F(\chi_1, \cdots, \chi_n) \in \mathcal{D}_{k_0}$, is called a mix-valued logical function.

Proposition 4.2 ([2]): Given a mix-valued logical function $F : \prod_{i=1}^{n} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$, assume $\chi_i \in \mathcal{D}_{k_i}$ has vector form $x_i \in \Delta_{k_i}$, $i = 1, \cdots, n$. Then, there exists a unique logical matrix $M_F \in \mathcal{L}_{k_0 \times k}$ ($k = \prod_{i=1}^{n} k_i$), such that

$$F(x_1, \cdots, x_n) = M_F \ltimes_{i=1}^{n} x_i, \quad x_i \in \Delta_{k_i}. \tag{39}$$

Definition 4.3: Assume $\chi \in \mathcal{D}_p$, $\xi \in \mathcal{D}_q$, where $p \neq q$, we define

$$\begin{cases} \chi \wedge \xi = \xi \wedge \chi = \chi, & \text{if } \xi = 1 \\ \chi \vee \xi = \xi \vee \chi = \chi, & \text{if } \xi = 0. \end{cases}$$

Splitting M_F into $t = k/k_n$ equal blocks as

$$M_f = [L^{1,\cdots,1}, \cdots, L^{1,\cdots,k_{n-1}}, \cdots, L^{k_1,k_2,\cdots,k_{n-1}}],$$

we define a set of unary operators $\phi^{i_1,i_2,\cdots,i_{n-1}} : \mathcal{D}_{k_n} \to \mathcal{D}_{k_0}$ with structure matrix $L^{i_1,i_2,\cdots,i_{n-1}}$, $i_s = 1, \cdots, k_s$, $s = 1, \cdots n-1$. Then we have the following normal form.

Theorem 4.4: Assume $F : \prod_{i=1}^{n} \mathcal{D}_{k_i} \to \mathcal{D}_{k_0}$ is an n-variable mix-valued logical function with algebraic form

$$F(x_1, \cdots, x_n) := M_F \ltimes_{i=1}^{n} x_i, \tag{40}$$

where $M_F \in \mathcal{L}_{k_0 \times k}$ is the structure matrix of F.

- Disjunctive normal form:

$$F(\chi_1, \cdots, \chi_n) = \bigvee_{i_1=1}^{k_1} \bigvee_{i_2=1}^{k_2} \cdots \bigvee_{i_{n-1}=1}^{k_{n-1}} \Big[\triangleleft_{k_1}^{i_1}(\chi_1) \vee \triangleleft_{k_2}^{i_2}(\chi_2) \vee \cdots \vee \triangleleft_{k_{n-1}}^{i_{n-1}}(\chi_{n-1}) \vee \varphi^{i_1,\cdots,i_{k-1}}(\chi_n) \Big]. \tag{41}$$

- Conjunctive normal form:

Assume $\neg F(\chi_1, \cdots, \chi_n)$ has disjunctive normal form as (41). Then using De Morgan formula, the conjunctive normal form of $F(\chi_1, \cdots, \chi_n)$ can be obtained as

$$F(\chi_1, \ldots, \chi_n) = \bigwedge_{i_1=1}^{k_1} \bigwedge_{i_2=1}^{k_2} \cdots \bigwedge_{i_{n-1}=1}^{k_{n-1}} \Big[\triangleleft_{k_1}^{i_1}(\chi_1) \bigvee \triangleleft_{k_2}^{i_2}(\chi_2) \bigvee \cdots \bigvee \triangleleft_{k_{n-1}}^{i_{n-1}}(\chi_{n-1}) \bigvee \phi^{k_0+1-i_1,\ldots,k_0+1-i_{k-1}}(\chi_n) \Big]. \tag{42}$$

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Controllability of dynamic-algebraic mix-valued logical control networks

Main Results:

- ✓ For the dynamic-algebraic mix-valued logical control networks, several **lower dimensional** controllability matrices are defined, then new necessary and sufficient conditions for the controllability are presented as well.

I. The algebraic form of dynamic-algebraic mix-valued logical control networks (DAMLCNs)

Using matrix expression, we denote a logical domain by $\mathcal{D}_k = \{0, \frac{1}{k-1}, \frac{2}{k-1}, \dots, 1\}$. If $\frac{i}{k-1} \sim \delta_k^{k-i}, i = 1, 2, \dots, k$, then $\mathcal{D}_k \sim \Delta_k$.

Now we study the controllability of DAMLCNs. Consider the following dynamic equations:

$$\begin{cases} x_1(t+1) = f_1(u_1(t), \dots, u_m(t), x_1(t), \dots, x_n(t)), \\ x_2(t+1) = f_2(u_1(t), \dots, u_m(t), x_1(t), \dots, x_n(t)), \\ \quad \vdots \\ x_r(t+1) = f_r(u_1(t), \dots, u_m(t), x_1(t), \dots, x_n(t)), \end{cases} \tag{2a}$$

for $t \geq 0$ and the state variables of remainder $n-r$ satisfy the following algebraic logical equations at $t \geq 0$,

$$\begin{cases} \delta_{k_{r+1}}^{i_{r+1}} = f_{r+1}(x_1(t), x_2(t), \dots, x_n(t)), \\ \delta_{k_{r+2}}^{i_{r+2}} = f_{r+2}(x_1(t), x_2(t), \dots, x_n(t)), \\ \quad \vdots \\ \delta_{k_n}^{i_n} = f_n(x_1(t), x_2(t), \dots, x_n(t)), \end{cases} \tag{2b}$$

where $x_i(t) \in \Delta_{k_i}, i \in [1, n]$ and $u_j(t) \in \Delta_{s_j}, j \in [1, m]$ are states and control inputs, respectively. Let $k = \prod_{i=1}^{n} k_i, \tilde{k} = \prod_{i=1}^{r} k_i, k' = \prod_{i=r+1}^{n} k_i$, and $q = \prod_{j=1}^{m} s_j$. $f_i : \Delta_{kq} \to \Delta_{k_i}$ and $f_j : \Delta_k \to \Delta_{k_j}, i \in [1, r], j \in [r+1, n]$ are logical functions, respectively.

The systems (2) can be **rewritten** as:

$$\begin{cases} x_1(t+1) = f_1(u_1(t), \dots, u_m(t), x_1(t), \dots, x_n(t)), \\ \quad \vdots \\ x_r(t+1) = f_r(u_1(t), \dots, u_m(t), x_1(t), \dots, x_n(t)), \\ \delta_{k_{r+1}}^{1} = f_{r+1}(x_1(t), x_2(t), \dots, x_n(t)), \\ \quad \vdots \\ \delta_{k_n}^{1} = f_n(x_1(t), x_2(t), \dots, x_n(t)), \end{cases} \tag{3}$$

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I. The algebraic form of dynamic-algebraic mix-valued logical control networks (DAMLCNs)

The system (3) can be converted to the following form using STP :

$$\begin{cases} x^1(t+1) = \tilde{L}u(t)x^1(t)x^2(t), & \text{(4a)} \\ \delta_{k'}^1 = Gx^1(t)x^2(t), & \text{(4b)} \end{cases}$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_k$, $x^1(t) = \ltimes_{i=1}^{r} x_i(t) \in \Delta_{\tilde{k}}$, $x^2(t) = \ltimes_{i=r+1}^{n} x_i(t) \in \Delta_{k'}$, and $\tilde{L} = M_{f_1} * \cdots * M_{f_r} \in \mathcal{L}_{\tilde{k}\times kq}$ is called the transition matrix of (4a), $G = M_{f_{r+1}} * \cdots * M_{f_n} \in \mathcal{L}_{k'\times k}$, and $*$ is the Khari-Rao product [4].

For (4b), we split G into $\tilde{k}$ equal blocks as

$$G = [G_1 \; G_2 \; \cdots \; G_{\tilde{k}}],$$

where $G_i \in \mathcal{L}_{k'\times k'}$, $i \in [1, \tilde{k}]$. Denote by $\mathcal{I}^1$ the set of indices i such that $\delta_{k'}^1 \in \mathrm{Col}(G_i)$. Let $\mathcal{X}^1$ be the set of all canonical vectors $\delta_{\tilde{k}}^i$ with $i \in \mathcal{I}^1$. Based on $\mathcal{X}^1$, we define $\mathcal{C}^1 = \Delta_{\tilde{k}} \backslash \mathcal{X}^1$ as the set of all canonical vector $\delta_{\tilde{k}}^i$ for which no vector $\delta_{k'}^j$ can be found such that by assuming $x^1 = \delta_{\tilde{k}}^i$ and $x^2 = \delta_{k'}^j$, condition $(4b)$ holds. Moreover, denote by $\mathcal{I}_0^1$ the set of indices i such that only one of the columns of G_i coincides with $\delta_{k'}^1$ and let $\mathcal{X}_0^1$ be the set of canonical vectors $\delta_{\tilde{k}}^i$ with $i \in \mathcal{I}_0^1$. It is obvious that $\mathcal{X}_0^1 \subset \mathcal{X}^1$.

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I. The algebraic form of dynamic-algebraic mix-valued logical control networks (DAMLCNs)

Define a new matrix M_g of dimension $k' \times k$ with its entries being

M_g $$\begin{cases} (M_g)_{ji} = 1, \text{ if } \mathrm{Col}_j(G_i) = \delta_{k'}^1, & \text{(5a)} \\ (M_g)_{ji} = 0, \text{ if } \mathrm{Col}_j(G_i) \neq \delta_{k'}^1. & \text{(5b)} \end{cases}$$

For given $i \in \mathcal{I}^1$, denote the set of all j such that $(M_g)_{ji} = 1$ by $s(i)$. If $\sum_{j=1}^{k'}(M_g)_{ji} > 1$, then there are more than one element in $s(i)$. In fact, for any $j \in s(i)$, $x^2(t) = \delta_{k'}^j$ is the solution to (4b) corresponding to $x^1(t) = \delta_{\tilde{k}}^i$. Therefore, (4b) has multiple solutions x^2 when $\sum_{j=1}^{k'}(M_g)_{ji} > 1$. Assume $S(i) = \{\delta_{k'}^j : j \in s(i)\}$, then we define a set of matrices

$$S(M_g) = \{M : \mathrm{Col}_i(M) \in S(i) \text{ if } \sum_{j=1}^{k'}(M_g)_{ji} > 1, \\ \text{otherwise } \mathrm{Col}_i(M) = \mathrm{Col}_i(M_g), i \in Q_r\} \quad (6)$$

Therefore, equation $(4b)$ can be equivalently rewritten as

$$x^2(t) \in S(M_g)x^1(t), \;\; x^1(t) \in \mathcal{X}^1, \quad (7)$$

and system (3) is rewritten as

$$\begin{cases} x^1(t+1) = \tilde{L}u(t)x^1(t)x^2(t), \\ x^2(t) \in S(M_g)x^1(t), \; x^1(t) \in \mathcal{X}^1, \end{cases} \quad (8)$$

i.e.,

$$\begin{aligned} x^1(t+1) \in & \tilde{L}u(t)x^1(t)S(M_g)x^1(t) \\ = & \tilde{L}u(t)S(\bar{M}_g)x^1(t), \; x^1(t) \in \mathcal{X}^1, \end{aligned} \quad (9)$$

where $S(\bar{M}_g) = (I_{\tilde{k}} \otimes S(M_g))\Phi_{\tilde{k}}$, and $\Phi_{\tilde{k}} = \mathrm{Diag}\{\delta_{\tilde{k}}^1, \delta_{\tilde{k}}^2, \cdots, \delta_{\tilde{k}}^{\tilde{k}}\} \in \mathcal{L}_{\tilde{k}^2 \times \tilde{k}}$ is the power reducing matrix. Specially, when $\mathcal{I}^1 = \mathcal{I}_0^1$, then (9) can be rewritten as

$$x^1(t+1) = \tilde{L}u(t)\bar{M}_g x^1(t), \; x^1(t) \in \mathcal{X}^1, \quad (10)$$

where $\bar{M}_g = (I_{\tilde{k}} \otimes M_g)\Phi_{\tilde{k}}$.

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I. The algebraic form of dynamic-algebraic mix-valued logical control networks (DAMLCNs)

$$\begin{cases} (M_g)_{ji} = 1, \text{ if } \mathrm{Col}_j(G_i) = \delta_{k'}^1, & (5a) \\ (M_g)_{ji} = 0, \text{ if } \mathrm{Col}_j(G_i) \neq \delta_{k'}^1. & (5b) \end{cases}$$

$$S(M_g) = \{M : \mathrm{Col}_i(M) \in S(i) \text{ if } \sum_{j=1}^{k'} (M_g)_{ji} > 1, \text{ otherwise } \mathrm{Col}_i(M) = \mathrm{Col}_i(M_g), i \in Q_r\}.$$

Example 1. *Assume that* $G = \delta_3[1, 2, 3, 2, 2, 3, 1, 1, 1, 1, 2, 2]$, $x_1, x_2 \in \mathcal{D}_2, x_3 \in \mathcal{D}_3$ *and* $r = 2$, *then we have*

$$M_g = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore, $\mathcal{X}^1 = \{\delta_4^1, \delta_4^3, \delta_4^4\}$, $\mathcal{X}_0^1 = \{\delta_4^1, \delta_4^4\}$, $S(M_g) = \{\delta_3[1, 0, 1, 1], \delta_3[1, 0, 2, 1], \delta_3[1, 0, 3, 1]\}$. *Taking* $x = \delta_{12}^1 = \delta_4^1\delta_3^1$ *for instance, one gets* $M(\delta_{12}^1) = \{\delta_3[1, 0, 1, 1], \delta_3[1, 0, 2, 1], \delta_3[1, 0, 3, 1]\}$, *and* $x^2 = \delta_3^1 = Mx^1 = M\delta_4^1$ *for any* $M \in M(\delta_{12}^1)$.

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II. Controllability of DAMLCNs

The solution set of (4b)

Let $\mathcal{X}$ be the solution set of(4b), i.e., admissible state set of system (3). Since (7) is equivalent to (4b), we have

$$\mathcal{X} = \{x : x = x^1x^2 \in x^1S(M_g)x^1,\ x^1 \in \mathcal{X}^1\}.$$

Let $\mathcal{C} = \Delta_k \backslash \mathcal{X}$ and define $\mathcal{X}_0 = \{x : x = x^1x^2 \in x^1S(M_g)x^1, x^1 \in \mathcal{X}_0^1\}$. If solution set $\mathcal{X} = \emptyset$, then system (3) is obvious unsolvable. Hence we assume that $\mathcal{X} \neq \emptyset$ for a given mix-valued logical system.

Definition 3. *[4, 7] Consider system (3) with a given initial state $x(0) = x_0 \in \mathcal{X}$ and a final state $x_d \in \mathcal{X}_0$. x_d is said to be reachable from x_0 at the s-th step if there is a control sequence $U = \{u(0), u(1), \dots, u(s-1)\}$, such that $x(s) = x_d$. x_d is said to be reachable from x_0, if there exists $s > 0$, such that x_d is reachable from x_0 at the s-th step. System (3) is said to be reachable from x_0 if x_d is reachable from x_0, for every choice of $x_d \in \mathcal{X}_0$. System (3) is said to be controllable if x_d is reachable from x_0, for every choice of $x_0 \in \mathcal{X}$, $x_d \in \mathcal{X}_0$.*

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When $t = 0$, we have from (9) that

$$x^1(1) \in \tilde{L}u(0)S(\bar{M}_g)x^1(0) \text{ wtih } x^1(0) \in \mathcal{X}^1. \qquad (11)$$

Once $x(0) \in \mathcal{X}$ is fixed, there is a unique $x^2(0)$ corresponding to $x^1(0) \in \mathcal{X}^1$ such that $x(0) = x^1(0)x^2(0)$. It is learned from Definition 3 that the state sequence $x(t), 0 < t \leq s$, is determined uniquely by the system, which means that $x^1(1) \in \mathcal{X}_0^1$, then by definition of $M(x(0))$, $x^2(1) \equiv Mx^1(1)$ for any $M \in M(x(0)) \subset S(M_g)$. Consequently,

$$x^1(1) = \tilde{L}\tilde{M}u(0)x^1(0) \text{ with } x^1(0) \in \mathcal{X}^1, \qquad (12)$$

where $\tilde{M} = I_q \otimes \bar{M}$.

II. Controllability of DAMLCNs

As a result, for $t > 0$, it follows from (9) that

$$x^1(t+1) = \tilde{L}\tilde{M}u(t)x^1(t),\ x^1(t) \in \mathcal{X}^1. \qquad (13)$$

Denote $\bar{L} := \tilde{L}\tilde{M} \in M_{\tilde{k}\times\tilde{k}q}$, we split $\bar{L}$ in to q blocks as

$$\bar{L} = [Blk_1(\bar{L})\ Blk_2(\bar{L})\ \cdots\ Blk_q(\bar{L})], \qquad (14)$$

where $Blk_p(\bar{L}) \in \mathcal{L}_{\tilde{k}\times\tilde{k}}$, $p \in [1, q]$.

Assume that $\mathcal{I}_0^1 = \{\beta_1, \cdots, \beta_{s_1}\}$, $x^1(0) = \delta_{\tilde{k}}^i$, and for a given $s > 0$, $x^1(s) = \delta_{\tilde{k}}^j$. Let $W = \sum_{p=1}^{q} Blk_p(\bar{L})$. Denote $Blk_p(\bar{L}')$ by substituting zeros to the columns of $Blk_p(\bar{L})$ except for the i-th one and substituting zeros to the rows of $Blk_p(\bar{L})$ with indexes except $\beta_1, \cdots, \beta_{s_1}$. Then let $W_1 = \sum_{p=1}^{q} Blk_p(\bar{L}')$.

Similarly, let $Blk_p(\hat{L})$ be the matrix obtained from $Blk_p(\bar{L})$ by substituting zeros to the rows and columns of $Blk_p(\bar{L})$ with indexes except $\beta_1, \cdots, \beta_{s_1}$, then denote $\hat{W} = \sum_{p=1}^{q} Blk_p(\hat{L})$. Since W, W_1 and $\hat{W}$ depend on M, which are determined by a given $x(0)$, we use $W(M)$, $W_1(M)$ and $\hat{W}(M)$ instead of W, W_1 and $\hat{W}$, respectively.

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II. Controllability of DAMLCNs

Theorem 1. *Given* $x(0) = x_0 \in \mathcal{X}$, $x_d \in \mathcal{X}_0$ *and* $s > 0$ *such that* $x^1(0) = \delta_{\tilde{k}}^i$, $x^1(s) = \delta_{\tilde{k}}^j$ *and* $x^2(0) = Mx^1(0)$. *Consider system (3), and assume* $(\hat{W}(M)^{s-1}W_1(M))_{ji} = l$. *Then*

$$l = l(s; x(0), x_d) = (x^1(s))^T(\hat{W}(M)^{s-1}W_1(M))x^1(0). \quad (15)$$

Proof. We prove this by induction. When $s = 1$, let $u^1, \cdots, u^a$ be different control values that drive system (3) from $x(0)$ to $x(d) = x(1)$. From (12), for any $\lambda \in [1, a]$, we have

$$1 = (x^1(1))^T[Blk_1(\bar{L})\ Blk_2(\bar{L})\ \cdots\ Blk_q(\bar{L})]u^\lambda x^1(0).$$

Since each control value is a column of I_q, there exist $b = q - a$ different control value $v^1, v^2, \ldots, v^b$ such that

$$0 = (x^1(1))^T[Blk_1(\bar{L})\ Blk_2(\bar{L})\ \cdots\ Blk_q(\bar{L})]v^\kappa x^1(0),$$

where $\kappa \in [1, b]$. Summing up the above $a + b = q$ equations yields

$$\begin{aligned} a &= (x^1(1))^T[Blk_1(\bar{L})\ Blk_2(\bar{L})\ \cdots\ Blk_q(\bar{L})]\mathbf{1}_q^T x^1(0) \\ &= (x^1(1))^T W(M) x^1(0). \end{aligned} \quad (16)$$

It is learned from Definition 3 that there exist control sequences U such that $x^1(k) \in \mathcal{X}_0^1, k > 0$. Then we can get

$$a = (x^1(1))^T W_1(M) x^1(0),$$

which proves (15) for $s = 1$.

For the induction step, given $k > 0$, let $x^1(k) = \delta_{\tilde{k}}^{\beta_\tau} \in \mathcal{X}_0^1, \tau \in [1, s_1]$, and consider

$$\begin{aligned} &(x^1(k+1))^T W(M)^{k+1} x^1(0) \\ &= (x^1(k+1))^T W^k(M) \cdot W(M) x^1(0) \\ &= \sum_{\tau=1}^{s_1} (W(M)^k)_{j\beta_\tau} (W(M))_{\beta_\tau i} \\ &= \sum_{\tau=1}^{s_1} (\delta_{\tilde{k}}^j)^T W(M)^k \delta_{\tilde{k}}^{\beta_\tau} (\delta_{\tilde{k}}^{\beta_\tau})^T W(M) \delta_{\tilde{k}}^i. \end{aligned} \quad (17)$$

From the definitions of $W_1(M)$ and $\hat{W}(M)$, one can obtain that $(\delta_{\tilde{k}}^{\beta_\tau})^T W(M) \delta_{\tilde{k}}^i = (\delta_{\tilde{k}}^{\beta_\tau})^T W_1(M) \delta_{\tilde{k}}^i$ and $(\delta_{\tilde{k}}^j)^T W(M)^k \delta_{\tilde{k}}^{\beta_\tau}$ $= (\delta_{\tilde{k}}^j)^T \hat{W}(M)^k \delta_{\tilde{k}}^{\beta_\tau}$.

Using (17) we have

$$\begin{aligned} &(x^1(k+1))^T W(M)^{k+1} x^1(0) \\ &= \sum_{\tau=1}^{s_1} (\delta_{\tilde{k}}^j)^T \hat{W}(M)^k \delta_{\tilde{k}}^{\beta_\tau} (\delta_{\tilde{k}}^{\beta_\tau})^T W_1(M) \delta_{\tilde{k}}^i \\ &= \sum_{x^1(1) \in \mathcal{X}_0} (x^1(k+1))^T \hat{W}(M)^k x^1(1) (x^1(1))^T W_1(M) x^1(0). \end{aligned}$$

Applying the induction hypothesis yields

$$\begin{aligned} &(x^1(k+1))^T W(M)^{k+1} x^1(0) \\ &= \sum_{x^1(1) \in \mathcal{X}_0} l(k; x(1), x(k+1)) \cdot l(1; x(0), x(1)), \end{aligned}$$

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Theorem 1. *Given* $x(0) = x_0 \in \mathcal{X}$, $x_d \in \mathcal{X}_0$ *and* $s > 0$ *such that* $x^1(0) = \delta_{\tilde{k}}^i$, $x^1(s) = \delta_{\tilde{k}}^j$ *and* $x^2(0) = Mx^1(0)$. *Consider system (3), and assume* $(\hat{W}(M)^{s-1}W_1(M))_{ji} = l$. *Then*

$$l = l(s; x(0), x_d) = (x^1(s))^T(\hat{W}(M)^{s-1}W_1(M))x^1(0). \tag{15}$$

Applying the induction hypothesis yields

$$(x^1(k+1))^T W(M)^{k+1} x^1(0) = \sum_{x^1(1)\in\mathcal{X}_0} l(k; x(1), x(k+1)) \cdot l(1; x(0), x(1)),$$

which is exactly the number of control sequences that steer the system (3) from $x(0)$ to x_d in $k+1$ steps. The proof is completed. $\square$

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From Theorem 1 above, we have the following controllability criteria.

Theorem 2. *The following results hold for system (3),*

1) *Give* $x(0) \in \mathcal{X}$, $x_d \in \mathcal{X}_0$ *such that* $x^1(0) = \delta_{\tilde{k}}^i$, $x_d^1 = \delta_{\tilde{k}}^j$ *and* $x^2(0) = Mx^1(0)$. *Then* x_d *is reachable from* $x(0)$ *at the s-th step if and only if* $(\hat{W}(M)^{s-1}W_1(M))_{ji} > 0$.
2) *Given* $x(0)$, x_d *as 1).* x_d *is reachable from* $x(0)$ *if and only if* $\sum_{s=1}^{\tilde{k}} (\hat{W}(M)^{s-1}W_1(M))_{ji} > 0$.
3) *Given* $x(0)$ *as 1). System (3) is globally reachable from* $x(0)$ *if and only if* $\sum_{s=1}^{\tilde{k}} (\hat{W}(M)^{s-1}W_1(M))_{ji} > 0$, *for all* $j \in \mathcal{I}_0^1$.
4) *System (3) is globally controllable if and only if* $\sum_{s=1}^{\tilde{k}} (\hat{W}(M)^{s-1} W_1(M))_{ji} > 0$, *for all* $j \in \mathcal{I}_0^1$, $i \in \mathcal{I}^1$ *and* $M \in S(M_g)$.

Proof. 1) can be obtained directly from Theorem 1. By the Cayley-Hamilton Theorem, given an $\tilde{k} \times \tilde{k}$ matrix A, $A^{\tilde{k}}$ can be linear expressed by $I_{\tilde{k}}, A, A^2, \cdots, A^{\tilde{k}-1}$. Therefore, it is easy to see that if given $1 \leq i,j \leq \tilde{k}$, $((\hat{W}(M))^{s-1}W_1(M))_{ji} \equiv 0$, for any $1 \leq s \leq \tilde{k}$, then $(\hat{W}(M)^{s-1}W_1(M))_{ji} \equiv 0$, for any $s \in \mathbb{N}^+$. As a result, we only consider $\{\hat{W}(M)^{s-1}W_1(M) | 1 \leq s \leq \tilde{k}\}$. From the definition of controllability, x_d is reachable from $x(0)$ if and only if $\sum_{s=1}^{\infty} (\hat{W}(M)^{s-1}W_1(M))_{ji} > 0$, which is equivalent to $\sum_{s=1}^{\tilde{k}} (\hat{W}(M)^{s-1}W_1(M))_{ji} > 0$ from the analysis above. 3) can be deduced from 2) easily. From Definition 3, system (3) is globally controllable if and only if 3) holds for $x(0) \in \mathcal{X}$. Therefore, 4) holds. □

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III. Example

Example 2. *Considering the following system with three n-odes,*

$$\begin{cases} x_1(t+1) = f_1(x_1(t), x_2(t), x_3, u(t)), \\ x_2(t+1) = f_2(x_1(t), x_2(t), x_3, u(t)), \\ \delta_2^1 = f_3(x_1(t), x_2(t), x_3), \end{cases} \tag{18}$$

The truth table of systems (18) are listed as follow :

f_3	x_1	x_2	x_3
1	0	0	0
0	0	0	1
1	0	1	0
1	0	1	1
1	1	0	0
0	1	0	1
0	1	1	0
1	1	1	1

Table 2: Truth tables of f_3 in (18)

f_1	f_2	u	x_1	x_2	x_3
0	1	0	0	0	0
0	1	0	0	0	1
1	0	0	0	1	0
0	1	0	0	1	1
1	0	0	1	0	0
0	0	0	1	0	1
0	1	0	1	1	0
1	1	0	1	1	1
0	0	0.5	0	0	0
1	1	0.5	0	0	1
1	0	0.5	0	1	0
1	0	0.5	0	1	1
0	1	0.5	1	0	0
1	0	0.5	1	0	1
0	1	0.5	1	1	0
1	0	0.5	1	1	1
0	1	1	0	0	0
1	0	1	0	0	1
0	1	1	0	1	0
1	0	1	0	1	1
1	0	1	1	0	0
0	1	1	1	0	1
0	0	1	1	1	0
1	1	1	1	1	1

Table 1: Truth tables of $f_i, i \in [1, 2]$ in (18)

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III. Example

Using STP, the algebraic form of (18) is as follows:

$$\begin{aligned} x^1(t+1) =& \tilde{L}u(t)x^1(t)x^2(t), \\ \delta_2^1 =& Gx^1(t)x^2(t), \end{aligned} \quad (19)$$

where $\tilde{L} = \delta_4[3\ 3\ 2\ 3\ 2\ 4\ 3\ 1\ 4\ 1\ 2\ 2\ 3\ 2\ 3\ 2\ 3\ 2\ 3\ 2\ 2\ 3\ 4\ 1]$ *and* $G = \delta_2[1\ 2\ 1\ 1\ 1\ 2\ 2\ 1]$.

One gets $\mathcal{X}^1 = \{\delta_4^1, \delta_4^2, \delta_4^3, \delta_4^4\}$, $\mathcal{X}_0^1 = \{\delta_4^1, \delta_4^3, \delta_4^4\}$, $\mathcal{X} = \{\delta_8^1, \delta_8^3, \delta_8^4, \delta_8^5, \delta_8^8$, $\mathcal{X}_0 = \{\delta_8^1, \delta_8^5, \delta_8^8\}$, *and*

$$M_g = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then $S(M_g) = \{\delta_2[1,1,1,2], \delta_2[1,2,1,2]\}$, *and* $S(\bar{M}_g) = \{\delta_8[1,3,5,8], \delta_8[1,4,5,8]\}$.

Split L *into 3 blocks as* $L = [L_1\ L_2\ L_3]$, *where* $L_i \in \mathcal{L}_{2^2\times 2^3}$, $i \in [1,3]$. *When* $i \in [1,3]$, $L_i S(\bar{M}_g)x^1(t) \subsetneq \mathcal{X}_0^1, \forall x^1(t) \in \mathcal{X}^1$. *Therefore, the solution to (18) is not unique.*

Let $x(0) = \delta_8^1 = \delta_4^1\delta_2^1$, *one gets* $M = \delta_2[1,1,1,2]$. *Therefore,* $\bar{M} = \delta_8[1,3,5,8]$, *then* $\tilde{M} = I_3 \otimes \bar{M}$. *Split* $\tilde{L}\tilde{M}$ *into 3 blocks as* $\tilde{L}\tilde{M} = [\tilde{L}_1\ \tilde{L}_2\ \tilde{L}_3]$, *then* $\bar{L}_p'$ *and* $\hat{L}_p$, $p \in [1,3]$, *can be obtained, respectively. Therefore,*

$$W_1(M) = \sum_{p=1}^{3} \bar{L}_p' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}, \qquad \hat{W}(M) = \sum_{p=1}^{3} \hat{L}_p = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

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III. Example

Using STP, the algebraic form of (18) is as follows:

$$x^1(t+1) = \tilde{L}u(t)x^1(t)x^2(t), \quad \delta_2^1 = Gx^1(t)x^2(t), \qquad (19)$$

where $\tilde{L} = \delta_4[3\ 3\ 2\ 3\ 2\ 4\ 3\ 1\ 4\ 1\ 2\ 2\ 3\ 2\ 3\ 2\ 3\ 2\ 3\ 2\ 2\ 3\ 4\ 1]$ *and* $G = \delta_2[1\ 2\ 1\ 1\ 1\ 2\ 2\ 1]$.

One gets $\mathcal{X}^1 = \{\delta_4^1, \delta_4^2, \delta_4^3, \delta_4^4\}$, $\mathcal{X}_0^1 = \{\delta_4^1, \delta_4^3, \delta_4^4\}$, $\mathcal{X} = \{\delta_8^1, \delta_8^3, \delta_8^4, \delta_8^5, \delta_8^8$, $\mathcal{X}_0 = \{\delta_8^1, \delta_8^5, \delta_8^8\}$, *and*

$$M_g = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then $S(M_g) = \{\delta_2[1,1,1,2], \delta_2[1,2,1,2]\}$, *and* $S(\bar{M}_g) = \{\delta_8[1,3,5,8], \delta_8[1,4,5,8]\}$.

Since

$$\sum_{s=1}^{2^2} \hat{W}(M)^{s-1}W_1(M) = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix},$$

then it is easy to obtain that $\sum_{s=1}^{2^2} (\hat{W}(M)^{s-1}W_1(M))_{43} = 0$, *which means* $x_d = \delta_8^8 = \delta_4^4\delta_2^2$ *is not reachable from* $x_0 = \delta_8^5 = \delta_4^3\delta_2^1$. *As a result, system (18) is not globally reachable from* $x(0)$ *by result 3) of Theorem 2.*

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End