Bayesian Game and its *Ex-ante* Agent Transformation

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Outline

Introduction

- Basic concepts of Bayesian games are introduced.
- several problems related to Bayesian Nash equlibrium (BNE) seeking of Bayesian games are investigated.
- First, a new transformation of Bayesian games that preserves potentiality was proposed, and the resulting games were called *ex-ante* agent games.
- Then, a sufficient and necessary condition for a Bayesian game to have an *ex-ante* agent potential game is provided

Introduction

- We also proved that a Bayesian game with 2 players is a Bayesian potential game (BPG) if and only if its *ex-ante* agent game is a potential game.
- An algebraic representation for *ex-ante* agent games was also presented. Moreover, a potential equation for an *ex-ante* agent game was developed. An algorithm was designed for the BNE seeking in Bayesian games.
- An algorithm was designed for the BNE seeking in Bayesian games.
- Finally, we demonstrated our method of game formulation and BNE seeking through a routing game with incomplete information.

Private vs. Public Information

- Up to this point, we have assumed that players know all relevant information about each other. Such games are known as games with complete information.
- Incomplete Information: Players have private information about something relevant to his decision making.
    - Incomplete information introduces uncertainty about the game being played.
- Imperfect Information: Players do not perfectly observe the actions of other players or forget their own actions.

Private vs. Public Information

- In many game theoretic situations, one agent is unsure about the payoffs or preferences of others
- Examples:
    - Auctions: How much should you bid for an object that you want, knowing that others will also compete against you?
    - Market competition: Firms generally do not know the exact cost of their competitors
    - Social learning: How can you leverage the decisions of others in order to make better decisions
    - Signaling games: How should you infer the information of others from the signals they send

Private vs. Public Information

- We would like to understand what is a game of incomplete information, a.k.a. Bayesian games.
- First, we would like to differentiate private vs. public information.
- Example: Batle of Sex (BoS): "Coordination Game" (public information) In Sequential BoS, all information is public, meaning everyone can see all the same information:

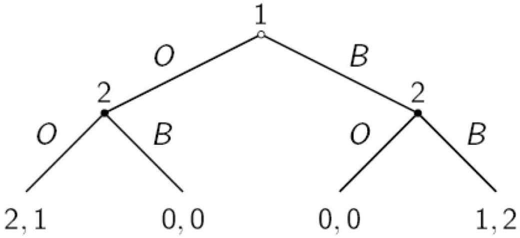
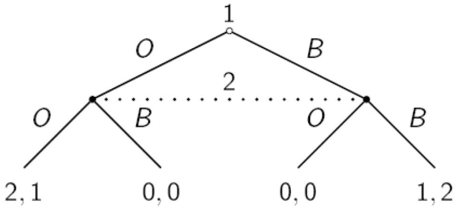


Fig. 1: 2 wishes to meet.

Private vs. Public Information

- In this extensive-form representation of regular BoS, Player 2 cannot observe the action chosen by Player 1.



- The previous is a game of imperfect information because players are unaware of the actions chosen by other player.
- They know who the other players are and their possible strategies/actions are. (public/complete information)
- Imagine that player 1 does not know whether player 2 wishes to meet or avoid player 1. (private/incomplete information)

Bayesian Games

- In games of incomplete information players may or may not know some information about the other players, e.g. their "type", their strategies, payoffs or preferences.
- Example: Tinder BoS Player 1 is unsure whether Player 2 wants to go out with her or avoid her, and thinks that these two possibilities are equally likely. Player 2 knows Player 1's preferences.

	O	B
O	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

2 wishes to meet

	O	B
O	$2, 0$	$0, 2$
B	$0, 1$	$1, 0$

2 wishes to avoid

- This is an example of a game in which one player does not know the payoffs of the other.

Bayesian Games

- More examples:
  - Bargaining over a surplus and you arent sure of the size
  - Buying a car of unsure quality
  - Job market: candidate is of unsure quality
  - Juries: unsure whether defendant is guilty
  - Auctions: sellers, buyers unsure of other buyers valuations
- When some players do not know the payoffs of the others, a game is said to have incomplete information. Its also known as a Bayesian game.

Bayesian Games

- Example: First-price auction (game with incomplete information)
  1. I have a copy of the Mona Lisa that I want to sell for cash
  2. Each of you has a private valuation for the painting, only known to you
  3. I will auction it off to the highest bidder
  4. Everyone submits a bid (sealed $\rightarrow$ simultaneous)
  5. Highest bidder wins the painting, pays their bid
  6. If tie, I will flip a coin

Bayesian Games

Definition 1

[3] A Bayesian game is a tuple $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$, where

- $\mathcal{N}$ is the finite player set;
- $\mathcal{A}_i$ is the action set of player i, $i \in \mathcal{N}$, and $\mathcal{A}$ is action profiles set;
- $\mathcal{T}_i$ is the finite type set of player i, $i \in \mathcal{N}$, and $\mathcal{T} = \prod_{i=1}^{n} \mathcal{T}_i$ is called the *type* profile set of $\mathcal{G}$;
- $p : \mathcal{T} \to [0, 1]$ is a probability distribution over $\mathcal{T}$, referred to as the common prior.
- $u = (u_i)_{i \in \mathcal{N}}$, where $u_i : \mathcal{T} \times \mathcal{A} \to \mathcal{R}$ is the utility function of player i, that maps each action profile $a \in \mathcal{A}$ to the utility of player i under each type profile $t \in \mathcal{T}$.

Bayesian Games

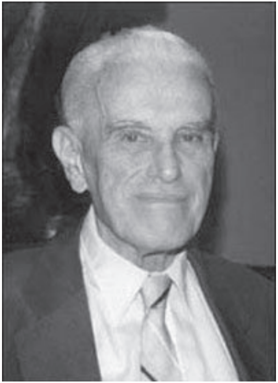


Figure 2: Nobel Memorial Prize in Economic Sciences: John C. Harsanyi/Robert J. Aumann/Reinhard Selten

Bayesian Games

- ▷ When players are not sure about the game they are playing you may consider:
- ❑ Player i needs to know what $-i$ knows about him. Also, Player i needs to know what i knows about $-i$. Moreover, i needs to know what $-i$ know about him conditional on what i know about $-i$, and so on

Fig. 3: https://v.qq.com/x/page/y323111r4s6.html.

Bayesian Games

Harsanyi (1968) doctrine [1]: There is a prior about the states of the nature that is common knowledge.

- Very strong assumption.
- But very convenient, because any private information is included in the description of the type and others can form beliefs about this type and each player understands others beliefs about his or her own type, and so on.
- Random events are considered an act of nature (that determine game structure)
- Treat nature as another (non-strategic) player
- Draw nature's decision nodes in extensive form

[1] J. C Harsanyi, Games with incomplete information played by Bayesian players part I. The basic model, Management Science, vol. 14, no. 3, pp.159-182, 1967.

Bayesian Games

Treat game as extensive form game with imperfect information:
Recall: BoS variant

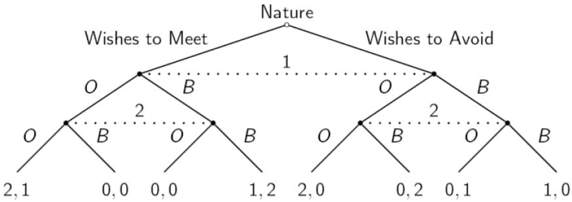
	O	B
O	$2,1$	$0,0$
B	$0,0$	$1,2$

2 wishes to meet

	O	B
O	$2,0$	$0,2$
B	$0,1$	$1,0$

2 wishes to avoid

Lets put this into extensive form:



Bayesian Games

Denote the *belief* of player i by

$$p(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)} = \frac{p(t_{-i}, t_i)}{\sum_{t_{-i} \in \mathcal{T}_{-i}} p(t_{-i}, t_i)}, \tag{1}$$

which describes player i's uncertainty about the other $n-1$ players' possible types t_{-i}, given player i's type t_i, where $\mathcal{T}_{-i} = \prod_{l \neq i} \mathcal{T}_l$ represents the set of the types of all the players except for the player i;

- Recall that player types are drawn from some prior probability distribution $p(t_{-i}, t_i)$.
- Given $p(t_1, ..., t_n)$ we can compute the conditional distribution $p(t_{-i}|t_i)$ using Bayes rule. Hence the label "Bayesian games.

Bayesian Games

Definition 2

[3] In Bayesian game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$, a strategy s_i for player $i \in \mathcal{N}$ is a mapping

$$s_i : \mathcal{T}_i \to \mathcal{A}_i,$$

which assigns to each type $t_i \in \mathcal{T}_i$ an action $a_i \in \mathcal{A}_i$. Denote S_i the set of strategies of player i. Denote the set of strategy profiles by

$$S = \prod_{i \in \mathcal{N}} S_i, \text{ with } S_i = (\mathcal{A}_i)^{\mathcal{T}_i}, \quad i \in \mathcal{N}.$$

Bayesian Games

Recall the battle of the sexes game,

	O	B
O	2, 1	0, 0
B	0, 0	1, 2

2 wishes to meet

	O	B
O	2, 0	0, 2
B	0, 1	1, 0

2 wishes to avoid

Then the strategies of player 1 and player 2 are
$S_1 = \{O, B\}$
$S_2 = \{\{O \text{ if type I}, O \text{ if type II}\}, \{O \text{ if type I}, B \text{ if type II}\},$
$\{B \text{ if type I}, B \text{ if type II}\}, , \{B \text{ if type I}, O \text{ if type II}\}\}$

Bayesian Games

Definition 3

[3] In a Bayesian game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$, the strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S$ is a pure Bayesian Nash equilibrium (BNE) if for each player $i \in \mathcal{N}$ and each type $t_i \in \mathcal{T}_i$, $s_i^*(t_i)$ solves

$$\max_{a_i \in \mathcal{A}_i} \sum_{t_{-i} \in T_{-i}} p(t_{-i}|t_i) u_j(t_{-i}, t_i; s_{-i}^*(t_{-i}), a_i), \tag{2}$$

Player i knows her own type and evaluates her expected payoffs according to the conditional distribution $p(t_{-i}|t_i)$.

Bayesian Games

Recall the battle of the sexes game,

- Let us consider the following strategy profile $(O, (O, B))$, which means that player 1 will play O, and while in type I, player 2 will also play O (when she wants to meet player 1) and in type II, player 2 will play B (when she wants to avoid player 1).
- Clearly, given the play of B by player 1, the strategy of player 2 is a best response.
- Let us now check that player 1 is also playing a best response. Since both types are equally likely, the expected payoff of player 1

$$E[O, (O, B)] = \frac{1}{2} \times 2 + \frac{1}{2} \times 0 = 1$$

- If, instead, he deviates and plays B, his expected payoff is

$$E[B, (O, B)] = \frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2}$$

- Therefore, the strategy profile $(O, (O, B))$ is a Bayesian Nash equilibrium.

Bayesian Games

- Interestingly, meeting at Beethoven' piano concert, which is the preferable outcome for player 2 is no longer a Nash equilibrium.
- Clearly, given the play of B by player 1, the strategy of player 2 is a best response.
- Suppose that the two players will meet at Football when they want to meet. Then the relevant strategy profile is $(B, (B, O))$ and

$$E[B, (B, O)] = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}$$

- If, instead, he deviates and plays B, his expected payoff is

$$E[O, (B, O)] = \frac{1}{2} \times 0 + \frac{1}{2} \times 2 = 1$$

- Therefore, the strategy profile $(B, (B, O))$ is not a Bayesian Nash equilibrium.

Bayesian Potential Games

Definition 4

[3] Given a Bayesian game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$, $\mathcal{G}$ is a Bayesian potential game (BPG) if there exists a function $\mathcal{F} : \mathcal{T} \times \mathcal{A} \to \mathcal{R}$, called potential function for $\mathcal{G}$, such that

$$u_i(t; a_i, a_{-i}) - u_i(t; a_i', a_{-i}) = \mathcal{F}(t; a_i, a_{-i}) - \mathcal{F}(t; a_i', a_{-i}),$$

for every $t \in \mathcal{T}$, $i \in \mathcal{N}$, $a_{-i} \in \mathcal{A}_{-i}$ and $a_i, a_i' \in \mathcal{A}_i$.

Bayesian Potential Games

An Example of Bayesian potential game

		t_{21}: a_{21}	t_{21}: a_{22}	t_{22}: a_{21}	t_{22}: a_{22}
t_{11}	a_{11}	0,0	1,2	1,0	1,0
	a_{12}	2,1	0,0	1,0	1,0
t_{12}	a_{11}	1,0	1,0	1,1	3,2
	a_{12}	0,1	0,1	3,4	2,2

Fig. 4: A Bayesian potential game.

Bayesian Potential Games

An Example of Bayesian potential game

		t_{21}		t_{22}	
		a_{21}	a_{22}	a_{21}	a_{22}
t_{11}	a_{11}	0	2	0	0
	a_{12}	2	1	0	0
t_{12}	a_{11}	0	0	0	1
	a_{12}	-1	-1	2	0

Fig. 5: A potential function $\mathcal{F}$.

Potential Games

Considering a normal form game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, u\}$, a typical 4-length simple closed path (SCP) $\phi = (\alpha, \beta, \gamma, \zeta)$ with two deviated players i, j is described in Fig. 6

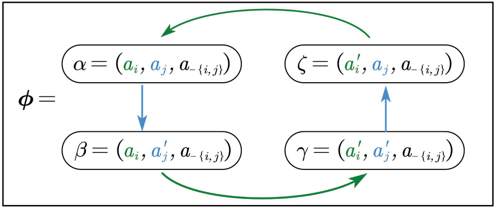


Fig. 6: A typical 4-length SCP in normal form game.

Potential Games

Let $\Gamma(\mathcal{A})$ denote the set of 4-length SCPs with two deviated players on the action profiles set $\mathcal{A}$ in the normal form game $\mathcal{G}$. Define the integral of utility function u along $\phi = (\alpha, \beta, \gamma, \zeta) \in \Gamma(\mathcal{A})$ as

$$I_u(\phi) = u_i(\beta) - u_i(\alpha) + u_j(\gamma) - u_j(\beta) + u_i(\zeta) - u_i(\gamma) + u_j(\alpha) - u_j(\zeta).$$

Lemma 5

[1] Let $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, u\}$ be a normal form game. Then, $\mathcal{G}$ is a deterministic potential game if and only if $I_u(\phi) = 0$ for all $\phi \in \Gamma(\mathcal{A})$.

Bayesian Potential Games

We give the following lemma to check whether a Bayesian game is potential, which is a direct consequence of Lemma 5.

Lemma 6

Let $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$ be a Bayesian game. Then, $\mathcal{G}$ is a BPG if and only if $\mathcal{I}_u(\phi, t) = 0$, for all $\phi \in \Gamma(\mathcal{A})$ and all $t \in \mathcal{T}$.

Ex-Ante Agent Games

Definition 7

An *ex-ante* agent game arising from the Bayesian game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$ is a tuple $\overline{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \overline{u}\}$, where

- $\overline{\mathcal{N}} = \{(i, t_{ij})\}_{i \in \mathcal{N}, t_{ij} \in \mathcal{T}_i}$ is the agent set;
- $\overline{\mathcal{A}} = \prod_{(i,t_{ij}) \in \overline{\mathcal{N}}} \mathcal{A}_{(i,t_{ij})}$ is the agent action profiles set;
- $\overline{u} = \left(\overline{u}_{(i,t_{ij})}\right)_{(i,t_{ij}) \in \overline{\mathcal{N}}}$, where $\overline{u}_{i,t_{ij}} : \overline{\mathcal{A}} \to \mathcal{R}$ is the utility function of agent $(i, t_{ij}) \in \overline{\mathcal{N}}$ defined as

$$\overline{u}_{i,t_{ij}}(\overline{a}) = \sum_{t_{-i} \in \mathcal{T}_{-i}} p(t_{-i}, t_{ij}) u_i(t_{-i}, t_{ij}; [\Phi^{-1}(\overline{a})](t_{-i}, t_{ij})). \tag{3}$$

Ex-Ante Agent Games

Definition 8

In the *ex-ante* agent game $\overline{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \overline{u}\}$, action profile $\overline{a}^* = (a^*_{(i,t_{ij})}, \overline{a}^*_{-(i,t_{ij})})$ is a pure NE of $\overline{\mathcal{G}}$ if for each $(i, t_{ij}) \in \overline{\mathcal{N}}$, $a^*_{(i,t_{ij})}$ solves

$$\max_{a_{(i,t_{ij})} \in \mathcal{A}_{(i,t_{ij})}} \overline{u}_{i,t_{ij}}(a_{(i,t_{ij})}, \overline{a}^*_{-(i,t_{ij})}). \tag{4}$$

Remark 1

It is noted that, if the agents set $\overline{\mathcal{N}}$ *and the action set* $\overline{\mathcal{A}}$ *remain unchanged in Definition 7, while the utility function* $\overline{u}_{i,t_{ij}}(\overline{a})$ *in (3) is replaced by*

$$\breve{u}_{i,t_{ij}}(\overline{a}) = \sum_{t_{-i} \in \mathcal{T}_{-i}} p(t_{-i}|t_{ij}) u_i(t_{-i}, t_{ij}; [\Phi^{-1}(\overline{a})](t_{-i}, t_{ij})), \tag{5}$$

then we get the interim agent game $\breve{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \breve{u}\}$ *proposed by Selten*

Potentiality preservation

The following theorem shows that the *ex-ante* agent game of a BPG is a (deterministic) potential game.

Theorem 9

Let $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$ be a BPG, then the ex-ante agent game $\overline{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \overline{u}\}$ of the game $\mathcal{G}$ is a deterministic potential game.

Sufficient and necessary condition of ex-ante potentiality

The two deviated agents in $\overline{\phi} \in \Gamma(\overline{\mathcal{A}})$ are either $((i, t_{ij}), (i, t_{ih})), j \neq h)$ from the same player $i \in \mathcal{N}$ or $((i, t_i^*), (k, t_k^*)) \in \overline{\mathcal{N}}$ from different players $i, k \in \mathcal{N}(i \neq k)$.

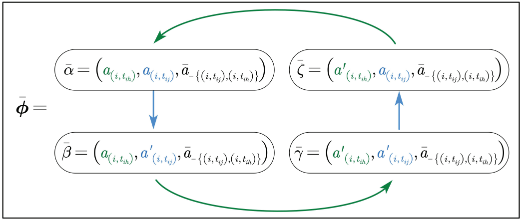


Fig. 7: A typical 4-length SCP with the 2 deviated agents$((i, t_{ij}), (i, t_{ih})) \in \overline{\mathcal{N}}$ from same player i.

Sufficient and necessary condition of ex-ante potentiality

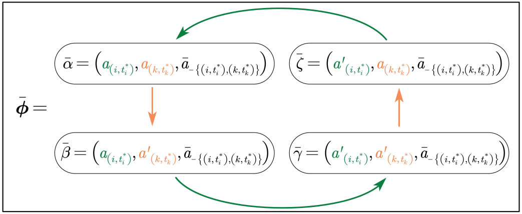


Fig. 8: A typical 4-length SCP with the 2 deviated agents $((i, t_i^*), (k, t_k^*)) \in \overline{\mathcal{N}}$ from different player i and k.

Sufficient and necessary condition of ex-ante potentiality

Lemma 10

Given a Bayesian game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$ *and its corresponding ex-ante agent game* $\overline{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \overline{u}\}$*. It holds that*

$$I_{\overline{u}}(\overline{\phi}) = 0, \quad \forall \overline{\phi} \in \Gamma_1(\overline{\mathcal{A}}). \tag{6}$$

where $\Gamma_1(\overline{\mathcal{A}}) = \bigcup_{i \in \mathcal{N}} \bigcup_{j \neq h} \Gamma(\overline{\mathcal{A}}, (i, t_{ij}), (i, t_{ih}))$.

Sufficient and necessary condition of ex-ante potentiality

Proposition 1

Define an operator Θ *from a 4-length path of ex-ante agent game* $\overline{\mathcal{G}}$ *to a 4-length path of Bayesian game* $\mathcal{G}$ *by specifying type* $t \in \mathcal{T}$*, as*

$$\Theta_t(\overline{\phi}) = \Big([\Phi^{-1}(\overline{a}^1)](t), [\Phi^{-1}(\overline{a}^2)](t), [\Phi^{-1}(\overline{a}^3)](t), [\Phi^{-1}(\overline{a}^4)](t)\Big), \quad (7)$$

for any given $\overline{\phi} = (\overline{a}^1, \overline{a}^2, \overline{a}^3, \overline{a}^4)$ *with* $\overline{a}^h \in \overline{\mathcal{A}}, h \in [1;4]$*. Then, for all given* $t_i^* \in \mathcal{T}_i, t_k^* \in \mathcal{T}_k$ *with* $i \neq k \in \mathcal{N}$*, we have*

$$\Theta_{(t_i^*, t_k^*, t_{-\{i,k\}})}(\overline{\phi}) \in \Gamma(\mathcal{A}, i, k), \quad (8)$$

for all $\overline{\phi} \in \Gamma(\overline{\mathcal{A}}, (i, t_i^*), (k, t_k^*))$*, and* $t_{-\{i,k\}} \in \mathcal{T}_{-\{i,k\}}$*.*

Sufficient and necessary condition of ex-ante potentiality

Lemma 11

For any $i \neq k \in \mathcal{N}$ *and arbitrary fixed* $t_i^* \in \mathcal{T}_i, t_k^* \in \mathcal{T}_k$*, the integral of utility function* $\overline{u}$ *along each* $\overline{\phi} \in \Gamma(\overline{\mathcal{A}}, (i, t_i^*), (k, t_k^*))$ *of the ex-ante agent game* $\overline{\mathcal{G}}$ *is*

$$I_{\overline{u}}(\overline{\phi}) = \sum_{t_{-\{i,k\}} \in \mathcal{T}_{-\{i,k\}}} \mathcal{I}_u \left(\Theta_{(t_i^*, t_k^*, t_{-\{i,k\}})}(\overline{\phi}), (t_i^*, t_k^*, t_{-\{i,k\}}) \right). \tag{9}$$

Sufficient and necessary condition of ex-ante potentiality

Theorem 12

Let $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$ be a Bayesian game. Then, its corresponding ex-ante agent game $\overline{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \overline{u}\}$ is a deterministic potential game if and only if, for any $\overline{\phi} \in \Gamma_2(\overline{\mathcal{A}})$,

$$\sum_{t_{-\{i,k\}} \in \mathcal{T}_{-\{i,k\}}} \mathcal{I}_u \left(\Theta_{(t_i^*, t_k^*, t_{-\{i,k\}})}(\overline{\phi}), (t_i^*, t_k^*, t_{-\{i,k\}}) \right) = 0, \tag{10}$$

where (i, t_i^*) and (k, t_k^*) is the two corresponding deviated agents of $\overline{\phi}$, and operator Θ is defined by (7).

Theorem 13

A Bayesian game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$ with 2 players is a BPG if and only if its ex-ante agent game $\overline{\mathcal{G}}$ is a potential game.

An *ex-ante* agent potential game arising from a non BPG with 3 players

Example 14

Consider a Bayesian game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$, where $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}_3 = \{1, 2\}, \mathcal{T}_1 = \{t_1\}, \mathcal{T}_2 = \{t_2\}, \mathcal{T}_3 = \{t_{31}, t_{32}\}, p(t_1, t_2, t_{31}) = 0.5, p(t_1, t_2, t_{32}) = 0.5$. The utility functions are shown in Table 1 and Table 2.

Table 1: The utility functions $u_i(t, a), i = 1, 2, 3$ for Example 14, where $t = (t_{11}, t_{21}, t_{31})$.

u/a	111	112	121	122	211	212	221	222
u_1	0	2	2	2	4	0	2	2
u_2	2	6	2	2	4	2	2	2
u_3	0	4	0	0	2	0	0	0

An *ex-ante* agent potential game arising from a non BPG with 3 players

By Definition 7, we can get its corresponding *ex-ante* agent game $\overline{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \overline{u}\}$, where $\overline{\mathcal{N}} = \{(1, t_{11}), (2, t_{21}), (3, t_{31}), (3, t_{32})\}$, $\mathcal{A}_{(1,t_{11})} = \mathcal{A}_{(2,t_{21})} = \mathcal{A}_{(3,t_{31})} = \mathcal{A}_{(3,t_{32})} = \{1, 2\}$.

Table 2: The utility functions $u_i(t, a), i = 1, 2, 3$ for Example 14, where $t = (t_{11}, t_{21}, t_{32})$.

u/a	111	112	121	122	111	212	221	222
u_1	4	6	2	2	4	0	2	2
u_2	2	6	2	2	4	2	2	2
u_3	0	4	0	0	2	0	0	0

Considering the 4 length SCP $\phi_1 = (\underline{12}1, \underline{11}1, \underline{21}1, \underline{22}1)$ under type profile $t = (t_{11}, t_{21}, t_{31})$, we find $\mathcal{I}_u(\phi_1, (t_{11}, t_{21}, t_{31})) = 1 \neq 0$. By Lemma 6, we get that the Bayesian game $\mathcal{G}$ is not a BPG.

The Semi-tensor Product of Matrices

Semi-tensor product of matrices[2]

Let $A \in \mathcal{M}_{m\times n}, B \in \mathcal{M}_{p\times q}$. The semi-tensor product of A and B, denoted as $A \ltimes B$, is defined by

$$A \ltimes B := (A \otimes I_{s/n})(B \otimes I_{s/p}),$$

where $s = lcm\{n, p\}$ is the least common multiple of n and p; $\otimes$ is the Kronecker product.

[2] D. Cheng, H. Qi, and Z. Li, Analysis and control of Boolean networks: a semi-tensor product approach. Springer, 2011.

Logical Network-based Approach

Lemma 15

[5] Let $f(x_1, x_2, \ldots, x_n) : \prod_{i=1}^{n} \Delta_{k_i} \to \mathcal{R}$ be a function. Then there exists a unique row vector $V_f \in \mathcal{R}_{1\times\prod_{i=1}^{n} k_i}$, called the structure vector of f, such that

$$f(x_1, x_2, \ldots, x_n) = V_f \ltimes_{i=1}^{n} x_i, \;\; x_i \in \Delta_{k_i}.$$

Corollary 16

Assume that f and g are two real-valued function on Δ_k with structure vector V_f and V_g, respectively, that is, $f(x) = V_f \ltimes x, g(x) = V_g \ltimes x$, for any $x \in \Delta_k$. Then,

$$\sum_{x\in\Delta_k} f(x)g(x) = V_f V_g^{\top}.$$

An algebraic representation and potential equations of ex-ante agent games

Proposition 2

Given an action profile $\overline{a} \in \overline{\mathcal{A}}$ *in the ex-ante agent game* $\overline{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \overline{u}\}$ *of the Bayesian game* $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$, *then the utility function of player* $i \in \mathcal{N}$ *in the game* $\mathcal{G}$, *under arbitrary fixed type profile* $t \in \mathcal{T}$ *is*

$$u_i(t; [\Phi^{-1}(\overline{a})](t)) = \mathbb{U} \ltimes \delta_n^i \ltimes \Upsilon(\overline{a}) \ltimes t, \;\; i \in \mathcal{N}, \tag{11}$$

where

$$\Upsilon(\overline{a}) = I_{c^n} \otimes \left[(\overline{a} \ltimes \delta_n^1) \otimes \cdots \otimes (\overline{a} \ltimes \delta_n^n) \right] M_{r,c^n}. \tag{12}$$

An algebraic representation of ex- ante agent games

Theorem 17

Given an action profile $\overline{a} \in \overline{\mathcal{A}}$ in the ex-ante agent game $\overline{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \overline{u}\}$ of the Bayesian game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$, then the utility of agent $(i, t_{ij}) \in \overline{\mathcal{N}}$ in the ex-ante agent game $\overline{\mathcal{G}}$ is

$$\overline{u}_{i,t_{ij}}(\overline{a}) = \mathbb{P} \ltimes \Lambda(i,j) \ltimes \Upsilon^\top(\overline{a}) \ltimes (\delta_n^i)^\top \ltimes \mathbb{U}^\top, \tag{13}$$

where $\Lambda(i,j) \in \mathcal{R}_{c^i \times c^i}(i \in \mathcal{N}, j \in [1; c])$ is defined as

$$Col_k \Lambda(i,j) := \begin{cases} \delta_{c^i}^k, & k = c(h-1) + j, h \in [1; c^{i-1}] \\ \delta_{c^i}^0, & else \end{cases}. \tag{14}$$

Potential equations of ex-ante agent games

Theorem 18

The ex-ante agent game $\overline{\mathcal{G}} = \{\overline{\mathcal{N}}, \overline{\mathcal{A}}, \overline{u}\}$ of the Bayesian game $\mathcal{G} = \{\mathcal{N}, \mathcal{A}, \mathcal{T}, p, u\}$ is a potential game, if and only if the ex-ante agent potential game equation

$$\Psi\xi = B\mathbb{P}^\top, \tag{15}$$

has a solution, where

$$\Psi = \begin{bmatrix} I_{b^{cn}} & \Psi^\top_{1,t_{11}} & 0 & \cdots & 0 \\ I_{b^{cn}} & 0 & \Psi^\top_{1,t_{12}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ I_{b^{cn}} & 0 & 0 & \cdots & \Psi^\top_{n,t_{nc}} \end{bmatrix}, \xi = \begin{bmatrix} \xi_0 \\ \xi_{1,t_{11}} \\ \xi_{1,t_{12}} \\ \vdots \\ \xi_{n,t_{nc}} \end{bmatrix},$$

$$B = [\mathcal{V}_{1,1}, \mathcal{V}_{1,2}, ..., \mathcal{V}_{1,c}, ..., \mathcal{V}_{n,1}, \mathcal{V}_{n,2}, ..., \mathcal{V}_{n,c}]^\top,$$

Potential equations of ex-ante agent games

where $\xi_0 \in \mathcal{R}_{b^{cn} \times 1}$ and $\xi_{i,t_{ij}} \in \mathcal{R}_{b^{cn-1} \times 1}, i \in [1; n], j \in [1; c]$, are unknowns.

Moreover, if the solution exists, the structure vector of the potential function $\overline{\mathcal{F}}$ is

$$V_{\overline{\mathcal{F}}} = \xi_0^\top.$$

Numerical Example

We consider a situation, corresponding to the traffic network in Fig. **??**.

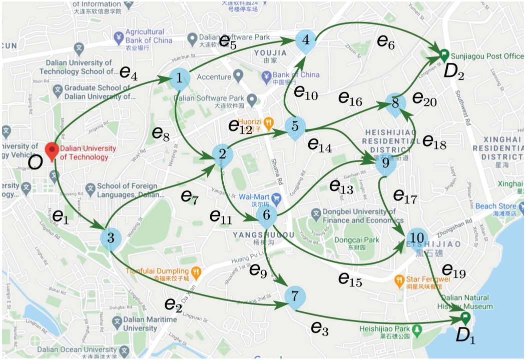


Fig. 9: A traffic network in the south of Dalian adopted from Google map, where $E = \{e_1, e_2, ..., e_{20}\}$ denotes the set of all edges in the network, O is Dalian University of Technology, D_1 is the Dalian Natural History Museum, D_2 is the Sunjiagou Post Office, and $V = \{1, 2, ..., 10\}$ denotes the set of all entry/exit points of the road edge, e.g., point 10 is the Heishijiao station.

Numerical Example

This network gives rise to a two-player Bayesian game. The private type of a player is its origin-destination pair. A simplified network $G = (V, E)$, shown by Fig. 10. As shown in Fig. 10, the paths from O to D_1 and O to D_2 are $\Xi_1, \Xi_2 = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8, \rho_9\}$.

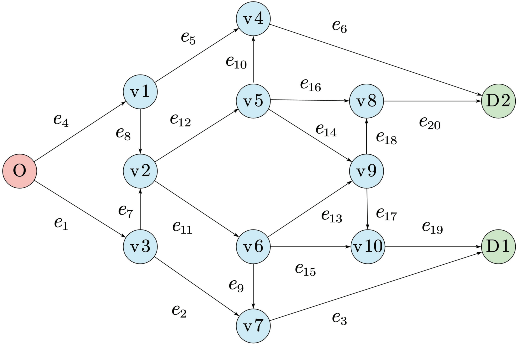


Fig. 10: Simplified diagram of the traffic network in Fig. 9.

Numerical Example

▷ The case without marginal-cost taxation

As illustrated in Fig. 11, the green dot line and purple solid line indicate the routines of agent $(1, t_{11})$ and agent $(1, t_{12})$ at $\bar{a}^*$, and the blue solid line and red dot line indicate the routines of agent $(2, t_{21})$ and agent $(2, t_{22})$ at $\bar{a}^*$.

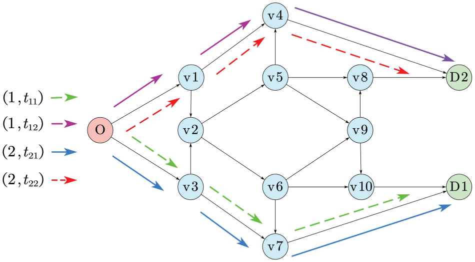


Fig. 11: The routes of agents at BNE without taxation.

Numerical Example

As observed, the action profile $\overline{a}^* = \overline{a}^1 = (\rho_1, \varrho_1, \rho_1, \varrho_1)$ satisfies $\overline{u}_{(i,t_{ij})}(a^*_{(i,t_{ij})}, \overline{a}^*_{-(i,t_{ij})}) \geq \overline{u}_{(i,t_{ij})}(a_{(i,t_{ij})}, \overline{a}^*_{-(i,t_{ij})})$, for any $a_{(i,t_{ij})} \in \mathcal{A}_{(i,t_{ij})}$. From the perspective of the definition of NE, we verify the correctness of the result.

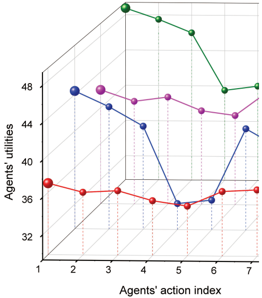


Fig. 12: Values of agents' utilities without taxation.

Numerical Example

▷ The case with marginal-cost taxes

As illustrated in Fig. 13, the green dot line and purple solid line indicate the routines of agent $(1, t_{11})$ and agent $(1, t_{12})$ at $\bar{a}^*$, and the blue solid line and red dot line indicate the routines of agent $(2, t_{21})$ and agent $(2, t_{22})$ at $\bar{a}^*$.

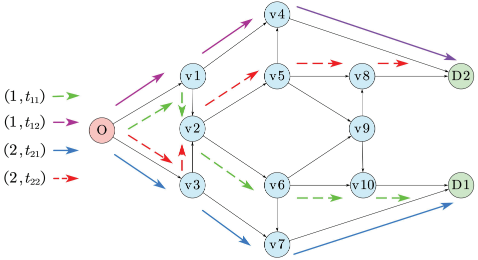


Fig. 13: The routes of agents at BNE with marginal-cost taxation.

Numerical Example

As observed, the action profile $\overline{a}^* = \overline{a}^1 = (\rho_1, \varrho_1, \rho_1, \varrho_1)$ satisfies $\overline{u}_{(i,t_{ij})}(a^*_{(i,t_{ij})}, \overline{a}^*_{-(i,t_{ij})}) \geq \overline{u}_{(i,t_{ij})}(a_{(i,t_{ij})}, \overline{a}^*_{-(i,t_{ij})})$, for any $a_{(i,t_{ij})} \in \mathcal{A}_{(i,t_{ij})}$. From the perspective of the definition of NE, we verify the correctness of the result.

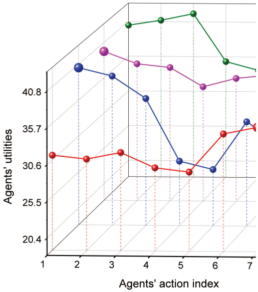


Fig. 14: Values of agents' utilities without taxation.

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Thank you for your attention !