研讨班

有限非合作博弈的空间分解

姓名: 郝亚琦

单位: 中科院系统所

导师: 张纪峰 教授

2021 年 7 月 20 日

Outline

Outline

1. Preliminaries

Definition 1.1

A (non-cooperative) finite normal form game $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ consists of three ingredients:

1) Players: $N := \{1, 2, \cdots, n\}$.

2) Strategy set: $S_i := \{1, 2, \cdots, k_i\}$, $i = 1, 2, \cdots, n$.
Strategy profile set: $S := \prod_{i=1}^{n} S_i$.
A strategy profile: $s = (s_1, s_2, \cdots, s_n) \in S$, where $s_i \in S_i$.

3) Utility function: $u_i : S \to \mathbb{R}$, $i = 1, 2, \cdots, n$.

$\mathcal{G}_{[n;k_1,\cdots,k_n]}$: the set of finite games with $|N| = n$, $|S_i| = k_i$, $i = 1, 2, \cdots, n$.

$S_{-i} := \prod_{j\neq i}^{n} S_j$: the strategy profile of players other than player i.

$s = (s_i, s_{-i})$, $u_i(s) = u_i(s_i, s_{-i})$.

1.1 Types of Games

$\mathcal{C}$: common interest game

$$u_i(s) = u_j(s), \quad \forall\, i, j;\ \forall\, s \in S.$$

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	a	b	c	d
u_2	a	b	c	d

$\in \mathcal{C}_{[2;2,2]}$

$\mathcal{Z}$: zero-sum game

$$\sum_{i=1}^{n} u_i(s) = 0, \quad \forall\, s \in S.$$

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	a	b	c	d
u_2	-a	-b	-c	-d

$\in \mathcal{Z}_{[2;2,2]}$

1.1 Types of Games

$\mathcal{L}$: normalized game   for ever $i \in N$ and every $s_{-i} \in S_{-i}$,

$$\sum_{s_i \in S_i} u_i(s_i, s_{-i}) = 0.$$

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	a	b	-a	-b
u_2	c	-c	d	-d

$\in \mathcal{L}_{[2;2,2]}$

$\mathcal{N}$: non-strategy game   for every $i \in N$ and every $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}), \quad \forall\, s_i, s_i' \in S_i.$$

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	a	b	a	b
u_2	c	c	d	d

$\in \mathcal{N}_{[2;2,2]}$

1.1 Types of Games

☞ Equivalence Relation

Definition 1.2

Let $G,\ \tilde{G} \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$. G and $\tilde{G}$ are said to be strategically equivalent, if for every $i \in N$, every $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = \tilde{u}_i(s_i, s_{-i}) - \tilde{u}_i(s'_i, s_{-i}), \forall\ s_i, s'_i \in S_i. \tag{1}$$

Lemma 1.1

The game G is strategically equivalent to game $\tilde{G}$, if and only if

$$G = \tilde{G} + N, \quad \textit{for some } N \in \mathcal{N}. \tag{2}$$

☞ Equivalence Relation

$$u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) = \tilde{u}_i(s_i, s_{-i}) - \tilde{u}_i(s'_i, s_{-i})$$

$\Leftrightarrow$

$$u_i(s_i, s_{-i}) - \tilde{u}_i(s_i, s_{-i}) = u_i(s'_i, s_{-i}) - \tilde{u}_i(s'_i, s_{-i}). \tag{3}$$

Let

$$d_i(s_i, s_{-i}) := u_i(s_i, s_{-i}) - \tilde{u}_i(s_i, s_{-i}). \tag{4}$$

$$(3) \Leftrightarrow d_i(s_i, s_{-i}) = d_i(s'_i, s_{-i}).$$

Define $\hat{G}$ with $\hat{u}_i(s_i, s_{-i}) = d_i(s_i, s_{-i}) \Rightarrow \hat{u}_i(s_i, s_{-i}) = \hat{u}_i(s'_i, s_{-i}) \Rightarrow \hat{G} \in \mathcal{N}$.

$$(4) \Leftrightarrow G = \tilde{G} + \hat{G},\ where\ \hat{G} \in \mathcal{N}$$

Example 1.1

Consider $G_1,\ G_2 \in \mathcal{G}_{[2;2,2]}$.

G_1:

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	a	b	c	d
u_2	e	f	g	h

G_2:

	(1,1)	(1,2)	(2,1)	(2,2)
$\tilde{u}_1$	A	B	C	D
$\tilde{u}_2$	E	F	G	H

Let $i = 1,\ s_1 = 1,\ s_1' = 2,\ s_{-1}(= s_2) = 1$.

$$u_1(1,1) - u_1(2,1) = \tilde{u}_1(1,1) - \tilde{u}_1(2,1).$$

$$\begin{aligned} a - c = A - C \quad &\Rightarrow \quad a - A = c - C := const(1,1) \\ &\Rightarrow \quad a = A + const(1,1), c = C + const(1,1). \end{aligned}$$

Example 1.1

Similarly,

$$b = B + const(1,2), d = D + const(1,2); e = E + const(2,1),$$
$$F = F + const(2,1), g = G + const(2,2), h = H + const(2,2).$$

G_1 and G_2 are strategically equivalent: $G_2 = G_1 + N$, where

N:

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	const(1,1)	const(1,2)	const(1,1)	const(1,2)
u_2	const(2,1)	const(2,1)	const(2,2)	const(2,2)

1.1 Types of Games

Notations	*Names*	*Definitions*
$\mathcal{Z}+\mathcal{N}$	*zero − sums equivalent game*	*strategically equivalent to a* $Z \in \mathcal{Z}$
$\mathcal{C}+\mathcal{N}$	*common interest equivalent games*	*strategically equivalent to a* $C \in \mathcal{C}$
$\mathcal{B}$	*zero − sum equivalent potential games*	*strategically equivalent to both a* $Z \in \mathcal{Z}$ *and a* $C \in \mathcal{C}$

Obviously, $\mathcal{B} = (\mathcal{Z}+\mathcal{N}) \cap (\mathcal{C}+\mathcal{N})$.

1.1 Types of Games

$$\begin{array}{|c|c|c|c|}\hline 3&2&1&6\\\hline 4&3&7&4\\\hline\end{array} = \underbrace{\begin{array}{|c|c|c|c|}\hline 1&-2&-1&2\\\hline -1&2&1&-2\\\hline\end{array}}_{zero-sum\ component} + \underbrace{\begin{array}{|c|c|c|c|}\hline 2&4&2&4\\\hline 5&5&6&6\\\hline\end{array}}_{non-strategy\ component} \in \mathcal{Z}+\mathcal{N}\setminus\mathcal{C}+\mathcal{N}.$$

$$\begin{array}{|c|c|c|c|}\hline 4&6&0&0\\\hline 3&4&-2&-3\\\hline\end{array} = \underbrace{\begin{array}{|c|c|c|c|}\hline 2&3&-2&-3\\\hline 2&3&-2&-3\\\hline\end{array}}_{common\ interest\ component} + \underbrace{\begin{array}{|c|c|c|c|}\hline 2&3&2&3\\\hline 1&1&0&0\\\hline\end{array}}_{non-strategy\ component} \in \mathcal{C}+\mathcal{N}\setminus\mathcal{Z}+\mathcal{N}.$$

$$\begin{array}{|c|c|c|c|}\hline 4&7&2&5\\\hline 6&5&8&7\\\hline\end{array} = \underbrace{\begin{array}{|c|c|c|c|}\hline 1&2&-1&0\\\hline -1&-2&1&0\\\hline\end{array}}_{zero-sum\ component} + \underbrace{\begin{array}{|c|c|c|c|}\hline 3&5&3&5\\\hline 7&7&7&7\\\hline\end{array}}_{non-strategy\ component} \in \mathcal{Z}+\mathcal{N}$$

$$= \underbrace{\begin{array}{|c|c|c|c|}\hline 4&3&2&1\\\hline 4&3&2&1\\\hline\end{array}}_{common\ interest\ component} + \underbrace{\begin{array}{|c|c|c|c|}\hline 0&4&0&4\\\hline 2&2&6&6\\\hline\end{array}}_{non-strategy\ component} \in \mathcal{C}+\mathcal{N}$$

$$\in \mathcal{B}$$

1.2 Potential Games

Definition 1.3

A game $G = (N, S, C) \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is called a potential game, if there exists a function $P : S \to \mathbb{R}$, such that for every $i \in N$, every $s_{-i} \in S_{-i}$,

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i}), \ \forall \ s_i, s_i' \in S_i. \tag{5}$$

The function P is called a potential function.

$\mathcal{G}^{\mathcal{P}}_{[n;k_1,\cdots,k_n]}$: $G \in \mathcal{G}^{\mathcal{P}} \Leftrightarrow G$ is strategically equivalent to a game $\tilde{G} = (N, S, \tilde{C})$, where

$$\tilde{u}_i(s) = P(s), \quad \forall \ s \in S, \quad i = 1, 2, \cdots, n. \tag{6}$$

$$\Updownarrow$$

$$\mathcal{G}^{\mathcal{P}} = \mathcal{C} + \mathcal{N}. \tag{7}$$

1.2 Potential Games

Example 1.1

Prison's Dilemma game: $G_{[2;2,2]}$, $S_1 = S_2 := \{1 := confess,\ 2 := defy\}$.

Table: Utility matrix of Prison's Dilemma Game

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	R	S	T	P
u_2	R	T	S	P

	(1,1)	(1,2)	(2,1)	(2,2)
P	R-T	0	0	P-S

G is a potential game.

$$\begin{bmatrix} R & S & T & P \\ R & T & S & P \end{bmatrix} = \underbrace{\begin{bmatrix} R-T & 0 & 0 & P-S \\ R-T & 0 & 0 & P-S \end{bmatrix}}_{common\ interest\ component} + \underbrace{\begin{bmatrix} T & S & T & S \\ T & T & S & S \end{bmatrix}}_{non-strategy\ component}$$

1.3 $\mathcal{G}_{[n;k_1,k_2,\cdots,k_n]} \cong \mathbb{R}^{nk}$

Consider a finite game $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N}) \in \mathcal{G}_{[n;k_1,k_2,\cdots,k_n]}$.

$$
\begin{aligned}
& u_1(s_1, \cdots, s_n) = V_1^u \ltimes s_i; \\
& u_2(s_1, \cdots, s_n) = V_2^u \ltimes s_i; \\
& \vdots \\
& u_n(s_1, \cdots, s_n) = V_n^u \ltimes s_i.
\end{aligned}
$$

Here $V_i^u \in \mathbb{R}^k$, $k = \times_{i=1}^n k_i$.
Set

$$
V_G = [V_1^u, V_2^u, \cdots, V_n^u] \in \mathbb{R}^{nk}.
$$

$$
\mathcal{G}_{[n;k_1,k_2,\cdots,k_n]} \cong \mathbb{R}^{nk}. \tag{8}
$$

1.4 Decomposition Results for Euclidean space

Let $\mathbb{R}^m$ be the Euclidean space with the standard inner product $< \cdot, \cdot >$: for two vectors $a = (a_1, \cdots, a_m)$, $b = (b_1, \cdots, b_m)$,

$$< a, b > := ab^{\mathrm{T}} = \sum_{i=1}^{m} a_i b_i. \quad (9)$$

Let $\mathcal{U},\ \mathcal{V} \in \mathbb{R}^m$ be two subspaces. $\mathcal{U} \cap \mathcal{V}$ is a closed subspace.

1) sum: $\mathcal{U} + \mathcal{V} := \{u + v \mid u \in \mathcal{U}, v \in \mathcal{V}\}$. $\mathcal{U} + \mathcal{V}$ is a closed subspace.

2) direct sum: $\mathcal{M} = \mathcal{U} \oplus \mathcal{V}$ if

- **(i)** $\mathcal{M} = \mathcal{U} + \mathcal{V}$;
- **(ii)** any $z \in \mathcal{M}$ can be uniquely written as the sum $z = u + v$ with $u \in \mathcal{U}, v \in \mathcal{V}$.

3) orthogonal complement (a canonical choice of $\mathcal{V}$):

$$\mathcal{V} = \mathcal{U}^{\perp} := \{v \in \mathbb{R}^m \mid < u, v > = 0,\ for\ all\ u \in \mathcal{U}\}.$$

1.4 Decomposition Results for Euclidean space

Lemma 1.2

Let $U, \; W \subseteq \mathbb{R}^m$ be two subspaces. Then

$$(U + W)^{\perp} = U^{\perp} \cap W^{\perp}. \tag{10}$$

Lemma 1.3

Let $U \subseteq \mathbb{R}^m$ be a closed subspace. Then

$$\mathbb{R}^m = U \oplus U^{\perp}. \tag{11}$$

Outline

2.1. $\mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}^*$

Inner product defined on $\mathcal{G}$: for G, $\hat{G} \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$,

$$\langle V_G, V_{\hat{G}} \rangle := V_G Q V_{\hat{G}}^T, \tag{12}$$

where

$$Q = \text{Diag}\left(\underbrace{k_1, \cdots, k_1}_{k}, \underbrace{k_2, \cdots, k_2}_{k}, \cdots, \underbrace{k_n, \cdots, k_n}_{k} \right) \in \mathcal{M}_{nk \times nk}.$$

$$\textit{game graph flow} \; + \; \textit{Helmholtz decomposition theorem}$$

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{}}_{\textit{Potential games}} \overbrace{\oplus \; \mathcal{H}}^{\textit{Harmonic games}}. \tag{13}$$

*O. Candogan, I. Menache, A. Ozdaglar, P.A. Parrilo, Flows and decompositions of games: Harmonic and potential games, *Mathematics of Operations Research*, 36(3), 474-503, 2011.

2.1. $\mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$

$\mathcal{P}$: pure potential game; $\mathcal{N}$: non-strategy game; $\mathcal{H}$: pure harmonic game.

$$\mathcal{G}^{\mathcal{P}} := \mathcal{P} \oplus \mathcal{N}; \quad \mathcal{G}^{\mathcal{H}} := \mathcal{H} \oplus \mathcal{N}.$$

	Harmonic Games	Potential Games
Subspaces	$\mathcal{H} \oplus \mathcal{N}$	$\mathcal{P} \oplus \mathcal{N}$
Pure NE	Generically does not exist	Always exists
Mixed NE	Uniformly mixed strategy is always a mixed NE	Always exists
Special Cases	$\mathcal{G}_{[2;k_1,k_2]}$: Set of mixed NE coincides with the set of correlated equilibria $\mathcal{G}_{[2;\kappa]}$: Uniformly mixed strategy is the unique mixed NE	$-$
Dynamics	Open questions	asynchronous learning

2.1. $\mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$

☞ Projection

Theorem 2.1.1

Let $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}) \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$, and let $\Pi_m = \delta_0^{\dagger} Du$. Then

1) The closest potential game to G has utilities $\prod_i \phi + (I - \prod_i) u_i$ for all $i \in N$;
2) The closest harmonic game to G has utilities $u_i - \prod_i \phi$ for all $i \in N$.

☞ Approximate equilibria

Theorem 2.1.2

Let $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$, and $\hat{G}$ be its closest potential game. Define $\alpha := \| G - \hat{G} \|_{[n;k_1,\cdots,k_n]}$. Then, every ϵ_1-equilibrium of $\hat{G}$ is an $\epsilon-$equilibrium of G for some $\epsilon \leq \max_{i \in N} \frac{2\alpha}{\sqrt{k_i}} + \epsilon_1$ (and vice versa).

2.2. $\mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$[*]

Inner product defined on $\mathcal{G}_{[n;k_1,\cdots,k_n]}$: for $G_1, G_2 \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$,

$$< G_1, G_2 > := V_{G_1} V_{G_2}^{\mathrm{T}}. \tag{14}$$

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}^{}}_{Potential\quad games}\!\!\!\!\!\!\!\overbrace{\phantom{\mathcal{N}} \oplus \mathcal{H}}^{Harmonic\quad games}.$$

[*]D. Cheng, T. Liu, K. Zhang, H. Qi, On decomposed subspaces of finite games, *IEEE Trans. Aut. Contr.*, 61(11), 3651-3656, 2016.

2.2. $\mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$[*]

Proposition 2.2.1

Consider a game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$. G is a pure potential game, if and only if for every $i \in N$, every $s_{-i} \in S_i$,

$$\sum_{s_i \in S_i} u_i(s_i, s_{-i}) = 0;$$

$$\exists P : S \to \mathbb{R}, \quad u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = P(s_i, s_{-i}) - P(s_i', s_{-i}). \qquad (15)$$

According to Proposition 2.3.2, it can be seen that

$$\mathcal{P} = (\mathcal{C} + \mathcal{N}) \cap \mathcal{L}. \qquad (16)$$

[*]T. Liu, H. Qi, D. Cheng, Dual expressions of decomposed subspaces of finite game, *Proceedings of the 34th Chinese Control Conference*, Hangzhou, 9146-9151, 2015.

Proposition 2.2.2

Consider a game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$. Then G is a pure harmonic game, if and only if

$$\sum_{i \in N} u_i(s) = 0, \quad \forall\, s \in S; \quad \sum_{s_i \in S_i} u_i(s_i, s_{-i}) = 0. \tag{17}$$

Then, it can be seen that

$$\mathcal{H} = \mathcal{Z} \cap \mathcal{L}. \tag{18}$$

$$\begin{aligned} \mathcal{P}^{\perp} &= [(\mathcal{C} + \mathcal{N}) \cap \mathcal{L}]^{\perp} \\ &= (\mathcal{C} + \mathcal{N})^{\perp} + \mathcal{L}^{\perp} \quad (\textit{Lemma } 2.4.1) \\ &= (\mathcal{C}^{\perp} \cap \mathcal{N}^{\perp}) + \mathcal{L}^{\perp} \quad (\textit{Lemma } 2.4.1) \\ &= (\mathcal{Z} \cap \mathcal{L}) + \mathcal{N} = (\mathcal{Z} \cap \mathcal{L}) \oplus \mathcal{N} \quad (\mathcal{L} \cap \mathcal{N} = \{\mathbf{0}\}). \end{aligned}$$

$$\mathcal{P} \textit{ is closed } + (\mathcal{L}^{\perp} = \mathcal{N}) \Rightarrow \mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}.$$

2.2. $\mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$

☞ Subspace

Define

$$k^{[p,q]} := \begin{cases} \Pi_{j=p}^{q} k_j, & q \geq p; \\ 1, & q < p. \end{cases}, \quad E_i = I_{k^{[1,i-1]}} \otimes \mathbf{1}_{k_i} \otimes I_{k^{[i+1,n]}}, \quad i = 1, 2, \cdots, n.$$

Theorem 2.2.1

1) Define $B_P = \begin{bmatrix} I_k - \frac{1}{k_1} E_1 E_1^{\mathrm{T}} \\ I_k - \frac{1}{k_2} E_2 E_2^{\mathrm{T}} \\ \vdots \\ I_k - \frac{1}{k_n} E_n E_n^{\mathrm{T}} \end{bmatrix}$. $\mathcal{P} = Span(B_P)$, which has $Col(B_P^0)$ as its basis, and B_P^0 is obtained by deleting any column of B_P.

2) Define $B_N = \begin{bmatrix} E_1 & 0 & 0 & \cdots & 0 \\ 0 & E_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & E_n \end{bmatrix}$. $\mathcal{N} = Span(B_N)$, which has $Col(B_N)$ as its basis.

Theorem 2.2.2

3) Define $B_H = [J_1, J_2, \cdots, J_{n-1}]$. $\mathcal{H} = Span(B_H)$, which has $Col(B_{H)})$ as its basis.

$$J_s = \begin{bmatrix} (\delta^1_{k_1} - \delta^{i_1}_{k_1})\delta^1_{k_2}\delta^1_{k_3}\cdots\delta^1_{k_s}(\delta^1_{k_{s+1}} - \delta^{i_{s+1}}_{k_{s+1}})\delta^{i_{s+2}}_{k_{s+2}}\cdots\delta^{i_n}_{k_n} \\ \delta^{i_1}_{k_1}(\delta^1_{k_2} - \delta^{i_2}_{k_2})\delta^1_{k_3}\cdots\delta^1_{k_s}(\delta^1_{k_{s+1}} - \delta^{i_{s+1}}_{k_{s+1}})\delta^{i_{s+2}}_{k_{s+2}}\cdots\delta^{i_n}_{k_n} \\ \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}(\delta^1_{k_3} - \delta^{i_3}_{k_3})\delta^1_{k_4}\cdots\delta^1_{k_s}(\delta^1_{k_{s+1}} - \delta^{i_{s+1}}_{k_{s+1}})\delta^{i_{s+2}}_{k_{s+2}}\cdots\delta^{i_n}_{k_n} \\ \vdots \\ \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\delta^{i_3}_{k_3}\cdots(\delta^1_{k_s} - \delta^{i_s}_{k_s})(\delta^1_{k_{s+1}} - \delta^{i_{s+1}}_{k_{s+1}})\cdots\delta^{i_n}_{k_n} \\ -(\delta^1_{k_1}\delta^1_{k_2}\cdots\delta^1_{k_s} - \delta^{i_1}_{k_1}\delta^{i_2}_{k_2}\cdots\delta^{i_s}_{k_s})(\delta^1_{k_{s+1}} - \delta^{i_{s+1}}_{k_{s+1}})\delta^{i_{s+2}}_{k_{s+2}}\cdots\delta^{i_n}_{k_n} \\ 0_{(n-1-s)k}, \end{bmatrix} (i_1, \cdots, i_s) \neq \mathbf{1}^{\mathrm{T}}_s,$$

$i_{s+1} \neq 1$, $s = 1, 2, \cdots, n-1$.

2.2. $\mathcal{G}=\mathcal{P}\oplus\mathcal{N}\oplus\mathcal{H}$

☞ Orthogonal projection

Theorem 2.2.3

Let $G\in\mathcal{G}_{[n;k_1,\cdots,k_n]}$. Assume its structure vector is

$$V_G=[V_1^c,V_2^c,\cdots,V_n^c].$$

Then

1) $\mathcal{G}^P:\ \pi_{\mathcal{G}^P}(G)=E_P^0\left((E_P^0)^TE_P^0\right)^{-1}(E_P^0)^{\mathrm{T}}\left(V_G\right)^{\mathrm{T}};$

2) $\mathcal{G}^N:\ \pi_{\mathcal{G}^N}(G)=B_N\left(B_N^TB_N\right)^{-1}B_N^{\mathrm{T}}\left(V_G\right)^{\mathrm{T}};$

3) $\mathcal{G}^{H_0}:\ \pi_{\mathcal{G}^{H_0}}(G)=B_H^0\left((B_H^0)^TB_H^0\right)^{-1}(B_H^0)^{\mathrm{T}}\left(V_G\right)^{\mathrm{T}};$

4) $\mathcal{G}^H:\ \pi_{\mathcal{G}^H}(G)=\pi_{\mathcal{G}^N}(G)+\pi_{\mathcal{G}^{H^0}}(G);$

5) $\mathcal{G}^{P_0}:\ \pi_{\mathcal{G}^{P_0}}(G)=\tilde{E}_P^0\left((\tilde{E}_P^0)^{\mathrm{T}}\tilde{E}_P^0\right)^{-1}(\tilde{E}_P^0)^{\mathrm{T}}\left(V_G\right)^{\mathrm{T}}.$

2.2. $\mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$

☞ Dynamically equivalent

Definition 2.2.1

Two evolutionary games are said to be dynamically equivalent, if they have the same strategy profile dynamics (that is, $f_i,\ i = 1, 2, \cdots, n,$).

$G \in \mathcal{G} \ \Rightarrow \ G_P := \pi_{\mathcal{G}^P}(G) \ \Rightarrow$ dynamically equivalent ?

If "yes", G has many good properties of the evolutionary potential game: existence and convergence of pure NEs, etc.

2.2. $\mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$

☞ Decomposition of network games

A network game $G^n = ((N, E), G)$: (N, E) is the network graph, G is the fundamental network game.
For $e \in E$, $G_e^{P_0}$: pure potential component; G_e^N: non-strategy component; $G_e^{H^0}$: pure harmonic component. Then

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P} \oplus \overbrace{\mathcal{N}}_{}}_{\mathcal{G}^{\mathcal{P}}} \oplus \mathcal{H}, \quad \overbrace{\mathcal{N} \oplus \mathcal{H}}^{\mathcal{G}^{\mathcal{H}}}.$$

Here

$$V_{G^{P_0}} = \sum_{e \in E} V_{G_e^{P_0}}, \quad V_{G^N} = \sum_{e \in E} V_{G_e^N}, \quad V_{G^{H_0}} = \sum_{e \in E} V_{G_e^{H_0}}.$$

2.2. $\mathcal{G} = \mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$

☞ Compare

1) $\mathcal{G}^N$ has $Col(B_N)$ as its basis, $\mathcal{G}^P$ has $Col(B_P^0)$ as its basis.

2) $\tilde{\mathcal{G}}^H$ has $Col(\tilde{B}_H)$ as its basis, where $\tilde{B}_H := Q^{-1}B_H$.

$\mathcal{G}^{\mathcal{H}}$ defined in [O. Candogan, 2009] $\neq \mathcal{G}^{\mathcal{H}}$ defined in [Cheng,2016].

2.3. $\mathcal{G} = \mathcal{P}^{\omega} \oplus \mathcal{N} \oplus \mathcal{H}^{\omega}$*

Definition 2.3.1

A game $G = (N, S, C) \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is called a weighted potential game, if there exists a function $P : S \to \mathbb{R}$, a set of weights $\{\omega_i\}_{i\in N}$, such that for every $i \in N$, every $s_{-i} \in S_{-i}$, and $\forall\ s_i, s_i' \in S_i$, we have

$$u_i(s_i, s_{-i}) - u_i(s_i', s_{-i}) = \omega_i[P(s_i, s_{-i}) - P(s_i', s_{-i})]. \tag{19}$$

$\mathcal{H}^{\omega} : V_G \in (\mathcal{P}^{\omega} \oplus \mathcal{N})^{\perp}$.

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P}^{\omega} \quad \oplus \quad \overbrace{\mathcal{N}}^{}}_{Weighted\ potential\ games}\!\!\!\!\!\!\!\!\overbrace{\phantom{\mathcal{N}} \quad \oplus \quad \mathcal{H}^{\omega}}^{Weighted\ harmonic\ games}. \tag{20}$$

*Y. Wang, T. Liu, D. Cheng, From weighted potential game to weighted harmonic game, *IET Control Theory Application*, 11(13), 2161-2169, 2017.

2.4. $\mathcal{G} = \mathcal{P}^{c\omega} \oplus \mathcal{N} \oplus \mathcal{H}^{c\omega}$*

Definition 2.4.1

A finite game $G = (N, S, C)$ is called coset weighted potential game, if there exists a function $P : S \to \mathbb{R}$, a set of coset weights $\{w_i(s_{-i}) \mid s_{-i} \in S_{-i},\ i = 1, 2, \cdots, n\}$, where $w_i(s_{-i})$ is independent of s_i, such that for every $i \in N$, every $s_{-i} \in S_{-i}$, and any $x,\ y \in S_i$,

$$u_i(x, s_{-i}) - u_i(y, s_{-i}) = w_i(s_{-i})[P(x, s_{-i}) - P(y, s_{-i})]. \tag{21}$$

$$\mathcal{G}_{[n;k_1,\cdots,k_n]} = \underbrace{\mathcal{P}^{c\omega} \oplus \overbrace{\mathcal{N}}}_{coset\ weighted\ potential\ games} \overbrace{\oplus \mathcal{H}^{c\omega}}^{coset\ weighted\ harmonic\ games}. \tag{22}$$

*Y. Wang, D. Cheng, On coset weighted potential game, *J. Franklin Inst.*, 357(9), 5523-5540, 2020.

Outline

3. Zero-sum Games and Potential Games*

Inner product defined on $\mathcal{G}$: for G, $\hat{G} \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$,

$$< G, \hat{G} >:= \sum_{i=1}^{n} \sum_{s_{-i}\in S_{-i}} \sum_{s_i \in S_i} (u_i(s_i, s_{-i})\hat{u}_i(s_i, s_{-i})) \,. \tag{23}$$

$$\Updownarrow$$

$$< G_1, G_2 >:= V_{G_1} V_{G_2}^{\mathrm{T}} \quad (\textit{standard inner product in } \mathbb{R}^{nk}). \tag{24}$$

$$\begin{aligned} \mathcal{G} &= \mathcal{Z} \oplus \mathcal{C}; \quad \mathcal{G} = \mathcal{L} \oplus \mathcal{N}; \\ \mathcal{G} &= (\mathcal{Z} \cap \mathcal{L}) \oplus (\mathcal{C} + \mathcal{N}); \\ \mathcal{G} &= (\mathcal{C} \cap \mathcal{L}) \oplus (\mathcal{Z} + \mathcal{N}); \\ \mathcal{G} &= (\mathcal{C} \cap \mathcal{L}) \oplus (\mathcal{Z} \cap \mathcal{L}) \oplus \mathcal{B}, \end{aligned}$$

where $\mathcal{B} = [(\mathcal{Z} + \mathcal{N}) \cap (\mathcal{C} + \mathcal{N})]$.

---

*S.H. Hwang, L. Rey-Bellet, Strategic decompositions of normal form games: zero-sum games and potential games, *Games and Economic Behavior*, 122, 370-390, 2020.

Example 3.1.1

Table: Utility matrix of $G \in \mathcal{G}_{[2;2,2]}$

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	a	b	c	d
u_2	e	f	g	h

$$\begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & g & h \\ \hline \end{array} = \underbrace{\begin{array}{|c|c|c|c|}\hline \frac{a+e}{2} & \frac{b+f}{2} & \frac{c+g}{2} & \frac{d+h}{2} \\ \hline \frac{a+e}{2} & \frac{b+f}{2} & \frac{c+g}{2} & \frac{d+h}{2} \\ \hline \end{array}}_{common\ interest\ component} + \underbrace{\begin{array}{|c|c|c|c|}\hline \frac{a-e}{2} & \frac{b-f}{2} & \frac{c-g}{2} & \frac{d-h}{2} \\ \hline \frac{e-a}{2} & \frac{f-b}{2} & \frac{g-c}{2} & \frac{h-d}{2} \\ \hline \end{array}}_{zero-sum\ component}$$

$$V_{C_{[2;2,2]}} = [\tfrac{a+e}{2}, \tfrac{b+f}{2}, \tfrac{c+g}{2}, \tfrac{d+h}{2}] \begin{bmatrix} 1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1 \end{bmatrix} (:= B_1);$$

$$V_{Z_{[2;,2,2]}} = [\tfrac{a-e}{2}, \tfrac{b-f}{2}, \tfrac{c-g}{2}, \tfrac{d-h}{2}] \begin{bmatrix} 1&0&0&0&-1&0&0&0 \\ 0&1&0&0&0&-1&0&0 \\ 0&0&1&0&0&0&-1&0 \\ 0&0&0&1&0&0&0&-1 \end{bmatrix} (:= B_2).$$

3.1. $\mathcal{G} = \mathcal{Z} \oplus \mathcal{C} \backslash \mathcal{L} \oplus \mathcal{N}$

Example 3.1.1

$\mathcal{C}_{[2;2,2]} = Span(B_1^T),\ \mathcal{Z}_{[2;2,2]} = Span(B_2^T),\ rank(B_1^T) = rank(B_1^T) = 4.$

$$B_1 B_2^T = 0 \Rightarrow \mathcal{G}_{[2;2,2]} = \mathcal{C}_{[2;2,2]} \oplus \mathcal{Z}_{[2;2,2]}.$$

$$\begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & g & h \\ \hline \end{array} = \underbrace{\begin{array}{|c|c|c|c|}\hline \frac{a-c}{2} & \frac{b-d}{2} & \frac{c-a}{2} & \frac{d-b}{2} \\ \hline \frac{e-f}{2} & \frac{f-e}{2} & \frac{g-h}{2} & \frac{h-g}{2} \\ \hline \end{array}}_{normalized\ \ component} + \underbrace{\begin{array}{|c|c|c|c|}\hline \frac{a+c}{2} & \frac{b+d}{2} & \frac{a+c}{2} & \frac{b+d}{2} \\ \hline \frac{e+f}{2} & \frac{e+f}{2} & \frac{g+h}{2} & \frac{g+h}{2} \\ \hline \end{array}}_{non-strategy\ \ component}$$

$$\mathcal{G}_{[2;2,2]} = \mathcal{L}_{[2;2,2]} \oplus \mathcal{N}_{[2;2,2]}.$$

3.1. $\mathcal{G} = \mathcal{Z} \oplus \mathcal{C} \backslash \mathcal{L} \oplus \mathcal{N}$

Consider the following homogeneous linear systems:

$$x_1 + x_2 + \cdots + x_n = 0, \tag{25}$$

$$x_1 = x_2 = \cdots = x_n, \tag{26}$$

where $x_i \in \mathbb{R},\ i = 1, 2, \cdots, n$.

Lemma 2.1.1

Let V_1 be the solution space of (25), and V_2 the solution space of (26). Then we have

$$\mathbb{R}^n = V_1 \oplus V_2. \tag{27}$$

3.1. $\mathcal{G} = \mathcal{Z} \oplus \mathcal{C} \backslash \mathcal{L} \oplus \mathcal{N}$

$\{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$ is a basis of V_1, where

$$\alpha_i^{\mathrm{T}} := [1, 0, \cdots, 0, \underset{\substack{\uparrow \\ (i+1)-th}}{-1}, 0, \cdots, 0] \in \mathbb{R}^n.$$

$\{\beta\}$ is a basis of V_2, where

$$\beta^{\mathrm{T}} = [1, 1, \cdots, 1] \in \mathbb{R}^n.$$

$$\alpha_i \beta^{\mathrm{T}} = 0, \quad i = 1, 2 \cdots, n-1 \;\Leftrightarrow\; V_2 = V_1^{\perp}. \tag{28}$$

3.1. $\mathcal{G} = \mathcal{Z} \oplus \mathcal{C} \backslash \mathcal{L} \oplus \mathcal{N}$

$$\begin{cases} G \in \mathcal{Z} \Leftrightarrow \sum_{i=1}^{n} u_i(s) = 0, \quad \forall\, s \in S \\ G \in \mathcal{C} \Leftrightarrow u_i(s) = u_j(s), \quad \forall\, s \in S \end{cases} \xrightarrow{(28)} \mathcal{G} = \mathcal{Z} \oplus \mathcal{C}. \tag{29}$$

$$\begin{cases} G \in \mathcal{L} \Leftrightarrow \sum_{s_i \in S_i}^{n} u_i(s_i, s_{-i}) = 0, \ \forall\, s_{-i} \in S_{-i} \\ G \in \mathcal{N} \Leftrightarrow u_i(s_i, s_{-i}) = u_i(s_i', s_{-i}), \ \forall\, s_{-i} \in S_{-i} \end{cases} \xrightarrow{(28)} \mathcal{G} = \mathcal{L} \oplus \mathcal{N}. \tag{30}$$

3.2. $\mathcal{G} = (\mathcal{Z} \cap \mathcal{L}) \oplus (\mathcal{C} + \mathcal{N})$

Using (10), we have that

$$(\mathcal{C} + \mathcal{N})^{\perp} = \mathcal{C}^{\perp} \cap \mathcal{N}^{\perp} = \mathcal{Z} \cap \mathcal{L}.$$

$$\mathcal{G}^{\mathcal{P}} \textit{ is closed} \Rightarrow \ \mathcal{G} = (\mathcal{C} + \mathcal{N}) \oplus (\mathcal{Z} \cap \mathcal{L}).$$

Similarly, the following decomposition can be obtained:

$$(\mathcal{Z} + \mathcal{N})^{\perp} = \mathcal{Z}^{\perp} \cap \mathcal{N}^{\perp} = \mathcal{C} \cap \mathcal{L} \Rightarrow \mathcal{G} = (\mathcal{Z} + \mathcal{N}) \oplus (\mathcal{C} \cap \mathcal{L}). \qquad (31)$$

3.3. $\mathcal{G} = (\mathcal{C} \cap \mathcal{L}) \oplus (\mathcal{Z} \cap \mathcal{L}) \oplus \mathcal{B}$

$$
\begin{aligned}
\mathcal{B}^{\perp} &= [(\mathcal{Z} + \mathcal{N}) \cap (\mathcal{C} + \mathcal{N})]^{\perp} \\
&= (\mathcal{Z} + \mathcal{N})^{\perp} + (\mathcal{C} + \mathcal{N})^{\perp} \quad (\textit{Lemma } 2.4.1) \\
&= (\mathcal{Z}^{\perp} \cap \mathcal{L}^{\perp}) + (\mathcal{C}^{\perp} \cap \mathcal{N}^{\perp}) \quad (\textit{Lemma } 2.4.1) \\
&= (\mathcal{C} \cap \mathcal{L}) + (\mathcal{Z} \cap \mathcal{L}) \\
&= (\mathcal{C} \cap \mathcal{L}) \oplus (\mathcal{Z} \cap \mathcal{L}) \quad (\mathcal{Z} \cap \mathcal{C} = \{\mathbf{0}\}).
\end{aligned}
$$

Orthogonal (canonical) direct sum decomposition:

$$
\mathcal{G} = (\mathcal{C} \cap \mathcal{L}) \oplus (\mathcal{Z} \cap \mathcal{L}) \oplus \mathcal{B} \;\; (\mathcal{Z}^{\perp} = \mathcal{C}). \tag{32}
$$

3.4. Equilibrium characterizations

1) $\mathcal{Z}+\mathcal{N}$: there exists a special function whose minima coincide with NE, playing an analogous role to potential functions; there is always a unique NE for finite two-player zero-sum equivalent games.

2) $\mathcal{B}$: two-player games in this class generically possess a strictly dominant strategy (a player' utility depend only on his (her) own actions. This observation is also extended to $n-$player zero-equivalent potential games).

Type	Properties of NE	Example
$\mathcal{Z}+\mathcal{N}$	uniqueness of NE	quasi-Cournot game(C)
$\mathcal{B}$	Two-player games: dominant NE	Prisoner's Dilemma(F)
$\mathcal{Z}\cap\mathcal{L}$	Unique uniform mixed NE	Matching Pennies game(F)
$\mathcal{C}\cap\mathcal{L}$	Uniform mixed NE	Coordination game(F)

Outline

4.1. $\mathcal{G} = \mathcal{S} \bigoplus \mathcal{S}^{\perp}$*

Definition 4.1.1

A game $G \in \mathcal{G}_{[n;\kappa]}$ is called an ordinary symmetric game if for any permutation $\sigma \in \mathbf{S}_n$,

$$u_i(x_1, \cdots, x_n) = u_{\sigma(i)}(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)}), \; i = 1, \cdots, n. \tag{33}$$

$\mathcal{S}_{[n;\kappa]}$: the set of symmetric games.

Strategy multiplicity vector of $s = (s_1, \cdots, s_n)$:

$$\sharp(s) = [\sharp(s,1), \sharp(s,2), \cdots, \sharp(s,\kappa)] \in \mathbb{R}^{\kappa},$$

where $\sharp(s,i) := |\{s_j \mid s_j = i\}|, \; i = 1, 2, \cdots, \kappa.$

*C. Li, F. He, T. Liu, D. Cheng, Symmetry-based decomposition of finite games, *Science China Information Science*, 62, 012207, 2019.

4.1. $\mathcal{G} = \mathcal{S} \oplus \mathcal{S}^{\perp}$

Proposition 4.1.1

A game $G \in \mathcal{G}_{[n;\kappa]}$ is called an ordinary symmetric game, if and only if, for any $s_i \in S_i$, $s_j \in S_j$, and any $s_{-i} \in S_{-i}$, $s_{-j} \in S_{-j}$, if $s_i = s_j$ and $\sharp(s_{-i}) = \sharp(s_{-j})$, then

$$u_i(s_i, s_{-i}) = u_j(s_j, s_{-j}), \quad 1 \leq i, j \leq n. \tag{34}$$

Example 4.1.1

Table: Payoff matrix of a symmetric game in $\mathcal{S}_{[3;2]}$:

$c\backslash a$	111	112	121	122	211	212	221	222
c_1	a	b	b	d	c	e	e	f
c_2	a	b	c	e	b	d	e	f
c_3	a	c	b	e	b	e	d	f

4.1. $\mathcal{G} = \mathcal{S} \oplus \mathcal{S}^{\perp}$

$$\mathcal{G}_{[n;\kappa]} = \mathcal{S}_{[n;\kappa]} \oplus \mathcal{A}_{[n;\kappa]} (= \mathcal{S}^{\perp}_{[n;\kappa]}). \tag{35}$$

$$\mathcal{S}_{[2;\kappa]} = \underbrace{\mathcal{SP} \quad \oplus \quad \overbrace{\mathcal{SN}}^{}}_{\textit{Potential component}} \!\!\!\!\!\!\! \overbrace{\phantom{\mathcal{SN}} \quad \oplus \quad \mathcal{SH}}^{\textit{Harmonic component}} . \tag{36}$$

$$\mathcal{A}_{[2;\kappa]} = \underbrace{\mathcal{AP} \quad \oplus \quad \mathcal{AN}}_{\textit{Potential component}} \!\!\!\!\!\!\! \overbrace{\phantom{\mathcal{AN}} \quad \oplus \quad \mathcal{AH}}^{\textit{Harmonic component}} . \tag{37}$$

4.2. $\mathcal{G} = \mathcal{S} \oplus \mathcal{K} \oplus \mathcal{E}$*

Definition 4.2.1

Let $G \in \mathcal{G}_{[n;\kappa]}$.

1) G is called a skew-symmetric game if for any permutation $\sigma \in \mathbf{S}_n$,

$$u_i(x_1, \cdots, x_n) = sign(\sigma) u_{\sigma(i)}(x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)}). \tag{38}$$

$\mathcal{K}_{[n;\kappa]} \in \mathbb{R}^{n\kappa^n}$: the set of skew-symmetric games.

2) G is called an asymmetric game if its structure vector

$$V_G \in \left[\mathcal{S}_{[n;\kappa]} \bigcup \mathcal{K}_{[n;\kappa]}\right]^{\perp}. \tag{39}$$

$\mathcal{E}_{[n;\kappa]} \in \mathbb{R}^{n\kappa^n}$: the set of asymmetric games.

---

*Y. Hao, D. Cheng, On Skew-Symmetric Games, *Journal of the Franklin Institute*, 355(6), 3196-3220, 2017.

Example 4.2.2

Table: Payoff matrix of $G \in \mathcal{G}_{[2;2]}$

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	α	γ	ξ	λ
u_2	β	δ	η	μ

$\mathcal{K}_{[2;2]} : S_2 = \{id, (1,2)\}$. Let $\sigma = (1,2) \Rightarrow$ sign(σ)=-1.

$$u_1(s_1, s_2) = sign(\sigma) u_2(s_2, s_1) = -u_2(s_2, s_1).$$

$$(s_1, s_2) = (1,1) \Rightarrow \alpha = -\beta; \qquad (s_1, s_2) = (1,2) \Rightarrow \gamma = -\eta;$$
$$(s_1, s_2) = (2,1) \Rightarrow \xi = -\delta; \qquad (s_1, s_2) = (1,2) \Rightarrow \lambda = -\mu.$$

4.2. $\mathcal{G} = \mathcal{S} \oplus \mathcal{K} \oplus \mathcal{E}^{\perp}$

Theorem 4.2.1

1) If $n > \kappa + 1$, then

$$\mathcal{G}_{[n;\kappa]} = \mathcal{S}_{[n;\kappa]} \oplus \mathcal{E}_{[n;\kappa]}; \tag{40}$$

2) If $n \leq \kappa + 1$, then

$$\mathcal{G}_{[n;\kappa]} = \mathcal{S}_{[n;\kappa]} \oplus \mathcal{K}_{[n;\kappa]} \oplus \mathcal{E}_{[n;\kappa]}. \tag{41}$$

Particularly, $\mathcal{E}_{[2;\kappa]} = \{0\}$. $\Rightarrow \mathcal{G}_{[2;\kappa]} = \mathcal{S}_{[2;\kappa]} \oplus \mathcal{K}_{[2;\kappa]}$.

Example 4.2.2

Table: Payoff matrix of $G \in \mathcal{G}_{[2;2]}$

	(1,1)	(1,2)	(2,1)	(2,2)
u_1	α	γ	ξ	λ
u_2	β	δ	η	μ

$$\begin{array}{|c|c|c|c|}\hline \alpha & \gamma & \xi & \lambda \\ \hline \beta & \delta & \eta & \mu \\ \hline \end{array} = \underbrace{\begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline a & c & b & d \\ \hline \end{array}}_{symmetric\ component} + \underbrace{\begin{array}{|c|c|c|c|}\hline a' & b' & -c' & d' \\ \hline -a' & -c' & -b' & -d' \\ \hline \end{array}}_{skew-symmetric\ component},$$

where

$$a = \frac{\alpha+\beta}{2}, \quad b = \frac{\gamma+\eta}{2}, \quad c = \frac{\xi+\delta}{2}, \quad d = \frac{\lambda+\mu}{2};$$
$$a' = \frac{\alpha-\beta}{2}, \quad b' = \frac{\gamma-\eta}{2}, \quad c' = \frac{\xi-\delta}{2}, \quad d' = \frac{\lambda-\mu}{2}.$$

5.3. Other special games

1) Boolean game: $\mathcal{G}_{[n;2,\cdots,2]}$ (+ symmetry $\Rightarrow$ potential game) (Cheng, Automatica, 2018)
2) Weighted potential game: $\mathcal{G}^{\omega}_{\mathcal{P}}$ (calculate weights) (Cheng, 2021)
3) Symmetric potential game: $\mathcal{S} \cap \mathcal{G}^{\mathcal{P}}$ (Hao, 2019)
4) Budget-balanced potential game: $\mathcal{G}^{\mathcal{P}}_{b}$ (Hao, 2021)
5) Group-based potential game: $\mathcal{G}^{\mathcal{P}}_{g}$ (Li, IEEE TCNS, 2019)
6) Zero-Sum Potential Games: $\mathcal{Z} \cap \mathcal{G}^{\mathcal{P}}$ (Li,2020)
7) Incomplete-profile potential games: $\mathcal{G}^{\Omega}_{\mathcal{P}}$ (Zhang, FI, 2018)

Outline

6 Decompositions

$\mathcal{C} \oplus \mathcal{Z}$	$d(\mathcal{C}) = k;\ d(\mathcal{Z}) = (n-1)k$
$\mathcal{L} \oplus \mathcal{N}$	$d(\mathcal{L}) = nk - \sum_{i=1}^{n} \frac{k}{k_i};\ d(\mathcal{N}) = \sum_{i=1}^{n} \frac{k}{k_i}$
$(\mathcal{C} \cap \mathcal{L}) \oplus (\mathcal{Z} + \mathcal{N})$	$d(\mathcal{C} \cap \mathcal{L}) = \Pi_{i=1}^{n}(k_i - 1)$
$(\mathcal{Z} \cap \mathcal{L}) \oplus (\mathcal{C} + \mathcal{N})$	$d(\mathcal{Z} \cap \mathcal{L}) = (n-1)k - \sum_{i=1}^{n} \frac{k}{k_i} + 1$
$(\mathcal{C} \cap \mathcal{L}) \oplus (\mathcal{Z} \cap \mathcal{L}) \oplus \mathcal{B}$	open question
$\mathcal{P} \oplus \mathcal{N} \oplus \mathcal{H}$ $[(\mathcal{C} + \mathcal{N}) \cap \mathcal{L}] \oplus \mathcal{N} \oplus (\mathcal{Z} \cap \mathcal{L})$	$d(\mathcal{P}) = k - 1,\ d(\mathcal{N}) = \sum_{i=1}^{n} \frac{k}{k_i}$
$\mathcal{S}_{[n;\kappa]} \oplus \mathcal{A}_{[n;\kappa]}$	$d(\mathcal{S}) = \kappa\binom{n+\kappa-2}{n-1}$
$\mathcal{S}_{[n;\kappa]} \oplus \mathcal{K}_{[n;\kappa]} \oplus \mathcal{E}_{[n;\kappa]}$	$d(\mathcal{K}) = \kappa\binom{\kappa}{n-1}$

谢 谢 大 家!

请大家批评指正！