An Introduction to Dimension-Free Manifolds With Applications to Control Systems

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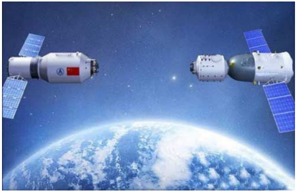
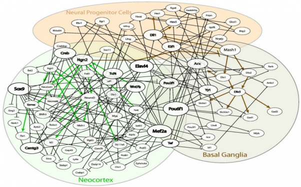
Background: Dimension-Varying Systems



(a) Spacecraft docking

(b) Vehicle clutch system

(c) Internet



(d) Genetic regulatory networks

Figure 1: Dimension-varying Systems

Outline of the course

- Basic notions and examples in differential topology
- Construction of dimension-free manifolds
- Application: dimension-varying control systems

Equivalence and Partitions

Definition 1.1

A binary relation $\sim$ over a set X is called an equivalence relation, if $\forall x, y, z \in X$,

1. (reflexive) $x \sim x$;
2. (symmetric) $x \sim y \Rightarrow y \sim x$;
3. (transitive) $x \sim y,\ y \sim z \Rightarrow x \sim z$.

Quotient Sets

Definition 1.2

Consider a set X and an equivalence $\sim$ on it. $\forall a \in X$, call a subset $[a] := \{x \in X \mid x \sim a\} \subset X$ an equivalence class with a representative element a. The set of all equivalence classes in X is called the quotient set of X with respect to equivalence $\sim$, denoted by $X/\sim$. The surjective map $\pi : X \to X/\sim$, $a \mapsto [a]$ is called the canonical map or quotient map.

Examples

Some examples of quotient sets from equivalences:

1. Let p be an integer. $\forall x, y \in \mathbb{Z}$, define $x \sim y \Leftrightarrow x = y \pmod{p}$, then it gives an equivalence on $\mathbb{Z}$, and the equivalence classes are the integers modulo p.
2. Let $p \in \mathbb{R}^n$, define

$$\mathcal{U}_p := \{U \in \mathbb{R}^n | U \text{ is an open set containing } p\}.$$

$\forall U \in \mathcal{U}_p$, denote by $C^\infty(U)$ all smooth functions over U, let $\mathcal{F} := \bigcup_{U \in \mathcal{U}_p} C^\infty(U)$, and define an equivalence on $\mathcal{F}$ by $\sim$: $\forall f, g \in \mathcal{F}$, for any $f \in C^\infty(U)$, $g \in C^\infty(V)$,

$$f \sim g \Leftrightarrow \exists W \in \mathcal{U}_p \text{ s.t. } W \subset U \cap V, f|_W = g|_W.$$

denote the quotient space by $C_p^\infty := \mathcal{F}/\sim$, which is called the germ of smooth functions at p.

A Special Class of Quotient Sets

Definition 1.3

Consider a set X and its subset S, define the following equivalence $\sim$: $\forall x, y \in X$, $x \sim y \Leftrightarrow \{x, y\} \subset S$. Denote the quotient set of X with respect to this equivalence as X/S, called the quotient set of X with respect to S.

Example: consider a finite automaton with observation, denoted by $\mathcal{A} = (X, Y, \Sigma, f, h, x_0, X_m)$. Define an equivalence over X as $x_1 \sim x_2 \Leftrightarrow h(x_1) = h(x_2)$, then the quotient space follows. Obviously, this is the quotient space with respect to $h^{-1}(y)$, $y \in Y$.

Topological Spaces

Definition 1.4

A topology $\mathcal{T}$ on a set X is a collection of subsets of X satisfying the following axioms:

(i) $\varnothing \in \mathcal{T}$, $X \in \mathcal{T}$.

(ii) $\forall O_1,\ O_2 \in \mathcal{T}$, $O_1 \cap O_2 \in \mathcal{T}$.

(iii) $\forall \{O_i\}_{i=1,2,\cdots} \subset \mathcal{T}$, $\bigcup_i O_i \in \mathcal{T}$.

The sets in $\mathcal{T}$ are called open sets; the complement of an open set is called a closed set.

Consider two topologies $\mathcal{T}_1$, $\mathcal{T}_2$ on X. If $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say $\mathcal{T}_2$ is finer (or bigger) than $\mathcal{T}_1$, and $\mathcal{T}_1$ is coarser (or smaller) than $\mathcal{T}_2$.

Euclidean Topology

Definition 1.5

Denote by d the Euclidean norm on $\mathbb{R}^n$, $p \in \mathbb{R}^n$, $r > 0$, define $B_r(p) := \{q \in \mathbb{R}^n | d(p, q) < r\}$, and define the open sets as arbitrary unions of $B_r(p)$, $\forall p \in \mathbb{R}^n$, $\forall r > 0$, the topology derived is called the Euclidean topology on $\mathbb{R}^n$.

Remark 1.1

A subset family $\mathcal{B} \subset \mathcal{T}$ is called the topological base of $(X, \mathcal{T})$, if every open set can be represented as the union of some sets in $\mathcal{B}$.

Continuous Maps and Homeomorphisms

Definition 1.6

Consider a map $f\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ between topological spaces. If $\forall O \in \mathcal{T}_2$, $f^{-1}(O) \in \mathcal{T}_1$, Then f is called a continuous map. If f is a bijection and its inverse is also continuous, it is called a homeomorphism between X_1 and X_2, and X_1, X_2 are called homeomorphic.

An open set containing x is called a neighbourhood of x. The above definition is equivalent to the following classical definition of continuity in function theory:

Definition 1.7

If $\forall x \in X_1$, for any neighbourhood V of $f(x)$ in X_2, there exist a neighbourhood $U \subset X_1$ of x, such that $f(U) \subset V$, then f is called a continuous map.

Topology From Mappings

Proposition 1.1

Consider a map $f\colon X \to (Y, \mathcal{T})$, let $\mathcal{T}$ be the topology on Y. define a collection of subsets over X consisting of the arbitrary union and finite intersection of the preimages of sets in $\mathcal{T}$ as

$$\mathcal{T}_f := \left\{ f^{-1}(U_1) \cap \cdots \cap f^{-1}(U_n) \,\middle|\, \{U_i\}_{i=1}^n \subset \mathcal{T}, n \in \mathbb{N} \right\} \cup \left\{ \bigcup_{\lambda\in\Lambda} f^{-1}(U_\lambda) \,\middle|\, \{U_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{T} \right\}$$

then $\mathcal{T}_f$ is the coarsest topology on X making the map f continuous, that is: for any topology $\mathcal{T}_X$ on X, if $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T})$ is continuous, then $\mathcal{T}_f \subset \mathcal{T}_X$.
Dually, for a map $g : (Y, \mathcal{T}) \to Z$, define the subset collection of Z as the arbitrary union and finite intersection of the images of sets in $\mathcal{T}$, denoted by $\mathcal{T}_g$, then $\mathcal{T}_g$ is the finest topology on Z making g continuous.

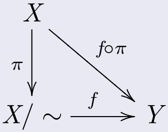
Quotient Topological Spaces

Definition 1.8

Consider a topological space $(X, \mathcal{T})$ and a quotient space $X/\sim$ of X. The finest topology on $X/\sim$ making the quotient map continuous is called the quotient topology on $X/\sim$, denoted by $\bar{\mathcal{T}}$. $(X/\sim, \bar{\mathcal{T}})$ is called the quotient topological space of X with respect to $\sim$.

Proposition 1.2

If X, Y are topological spaces and $X/\sim$ is the quotient space of X, let $f\colon X/\sim \to Y$ be any function, then f is continuous if and only if $f \circ \pi$ is continuous.



Examples of Quotient Spaces

1. The two-dimensional torus is the set $\mathbb{T}^2 := \mathbb{R}^2/\sim$, where $\sim$ is defined as: $x \sim y \Leftrightarrow x - y \in \mathbb{Z} \times \mathbb{Z}$.
2. The Möbius band is constructed as follows: consider a set $B : [0,1] \times (0,1) \subset \mathbb{R}^2$ endowed with Euclidean product, define $(0,a) \sim (1,-a)$, $\forall a \in (0,1)$. The corresponding quotient space $B/\sim$ is called the Möbius band.
3. Denote by $D^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leqslant 1\}$ the n-dimensional closed disk, let $S^{n-1} := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$ be the $n-1$-dimensional sphere, which is the boundary of D^n, then $D^n/S^{n-1} \simeq S^n$.

Smooth Manifolds

Definition 1.9

Let $(M, \mathcal{T})$ be a Hausdorff space with countable topological basis. If $\exists \{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$ satisfying $M = \bigcup_{\alpha \in I} U_\alpha$ and

1. $\forall \alpha \in I$, there exists open set $V_\alpha \subset \mathbb{R}^n$ and homeomorphism $\varphi_\alpha : U_\alpha \to V_\alpha$;
2. $\forall \alpha, \beta \in I$, if $U_\alpha \cap U_\beta \neq \varnothing$, then $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ and its inverse are both smooth maps,

then M is called an n-dimensional smooth manifold, $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is called a set of coordinates of M.

Tangent Vectors

Definition 1.10

Denote the germ of smooth functions at $p \in M$ as C_p^∞, let $v : C_p^\infty \to \mathbb{R}$ be a real function over C_p^∞, if $\forall [f], [g] \in C_p^\infty$, v satisfies

1. (Linearity) $\forall \alpha, \beta \in \mathbb{R}$, $v(\alpha[f] + \beta[g]) = \alpha v([f]) + \beta v([g])$;
2. (Leibniz rule) $v([f][g]) = f(p)v([g]) + g(p)v([f])$,

then v is called a tangent vector at p. The set of all tangent vectors at p is called the tangent space of M at p, denoted by T_pM.

Structure of Tangent Spaces

Given a coordinate $(U, \{y^i\}_{i=1}^n)$ of an n-dimensional manifold M, the tangent space T_pM can be viewed as the real vector space spanned by $\{\frac{\partial}{\partial y^i}\}_{i=1}^n$, where $\frac{\partial}{\partial y^i} : C_p^\infty \to \mathbb{R}$, $f \mapsto \frac{\partial f}{\partial y^i}|_p$ acts on a function by solving its partial derivative at p with respect to the i-th variable. Therefore, under this coordinate v can be represented as $v = \sum_{i=1}^n v_i \frac{\partial}{\partial y^i}$, $v(f) = \sum_{i=1}^n v_i \frac{\partial f}{\partial y^i}|_p$, where $v^i \in \mathbb{R}$, $i = 1, \cdots, n$.

Tangent Maps

Definition 1.11

Let $f\colon M \to N$ be a smooth map between smooth manifolds. Let $p \in M$, $q = f(p) \in N$, there is a map $f^* : C_q^\infty \to C_p^\infty$, $[g] \mapsto [g \circ f]$. Further, if $v \in T_pM$, then $v \circ f^* \in T_qM$, and f induces a map $f_* : T_pM \to T_qM$, $v \mapsto v \circ f^*$, called the tangent map of f at p.

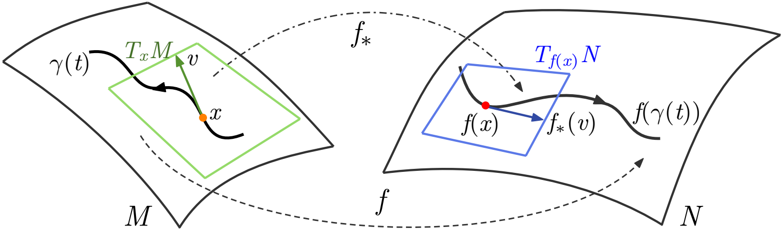


Figure 2: Tangent maps

Properties of Tangent Maps

Tangent maps can be viewed as infinitesimal characterization of smooth maps. Consider a smooth curve $\gamma : [0,1] \to M$ on the manifold M, its image under f is a smooth curve $f \circ \gamma : [0,1] \to N$ on N, let $a \in [0,1]$, $\gamma(a) = p$, $\dot{\gamma} = v \in T_pM$, then

$$f_*(v) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))|_{t=a}.$$

Proposition 1.3

Let M, N, O be smooth manifolds, $f, g : M \to N$, $h : N \to O$ be smooth maps, id_M be the identity map.

- $(\mathrm{id}_M)_*|_p = \mathrm{id}_{T_pM}$;
- $(h \circ g)_* = h_* \circ g_*$;
- $(\alpha f + \beta g)_* = \alpha f_* + \beta g_*$, $\forall \alpha, \beta \in \mathbb{R}$.

Smooth Vector Fields

Definition 1.12

Let M be a manifold, assigning a tangent vector at each point $p \in M$ yields a vector field over M, denoted by X; $X|_p \in T_pM$ represents the value X takes at p. If $\forall f \in C^\infty(M)$, $X(f) \in C^\infty(M)$, then call X a smooth vector field over M. The set of all smooth vector fields over M is denoted by $\mathfrak{X}(M)$.

Proposition 1.4

Let M be an n-dimensional manifold, then $X \in \mathfrak{X}(M)$ if and only if for any coordinate $(U, \{y^i\}_{i=1}^n)$, $X|_U = \sum_{i=1}^n f^i \frac{\partial}{\partial y^i}$, where f^i is a smooth function over U, $i = 1, \cdots, n$.

Integral Curves and Affine Control Systems

Definition 1.13

Let M be a manifold, $U \subset M$ is an open set, $X \in \mathfrak{X}(U)$, $\gamma : [a, b] \to U$ is a smooth map. If $\dot{\gamma}(t) := \gamma_*(\frac{\mathrm{d}}{\mathrm{d}t}) = X|_{\gamma(t)}$, then γ is called an integral curve of X.

Definition 1.14

Consider $f, g_1, \cdots, g_m \in \mathfrak{X}(M)$, the integral curve corresponding to

$$\dot{\gamma}_u(t) = f|_{\gamma_u(t)} + \sum_{i=1}^{m} u^i g_i|_{\gamma_u(t)}$$

is called an affine control system on M, where the bounded functions $u^i : [0, T] \to \mathbb{R}$ are called controls.

Pan-Dimensional State Space

We view all n-dimensional real Euclidean spaces as $\mathbb{R}^n$. Define the topological sum

$$\mathbb{R}^\infty := \bigsqcup_{n\in\mathbb{N}^+} \mathbb{R}^n$$

where each $\mathbb{R}^n$ possesses the n-dimensional Euclidean topology. Denote this natural topology on $\mathbb{R}^\infty$ as $\mathcal{T}_{\mathrm{n}}$.

Our aim is to make $\mathbb{R}^\infty$ a (topological) vector space.

Vector Addition

Definition 2.1

$\forall x \in \mathbb{R}^m \subset \mathbb{R}^\infty$, $y \in \mathbb{R}^n \subset \mathbb{R}^\infty$. Denote $\mathbf{1}_k := [\underbrace{1, \cdots, 1}_{k}]^{\mathrm{T}}$.

The vector addition (V-addition):

$$x \vec{+} y := \left(x \otimes \mathbf{1}_{t/m}\right) + \left(y \otimes \mathbf{1}_{t/n}\right) \in \mathbb{R}^\infty, \tag{1}$$

where $t = \mathrm{lcm}(m, n)$ is the least common multiple of m and n. Correspondingly, the substraction is defined as $x \vec{-} y := x \vec{+} (-y)$.

Inner Product

Definition 2.2

$\forall x \in \mathbb{R}^m \subset \mathbb{R}^\infty$, $y \in \mathbb{R}^n \subset \mathbb{R}^\infty$, define

1. Inner product (of x and y):

$$\langle x, y \rangle_{\mathcal{V}} := \frac{1}{t} \left\langle (x \otimes \mathbf{1}_{t/m}), (y \otimes \mathbf{1}_{t/n}) \right\rangle . \qquad (2)$$

2. Norm (of x): $\|x\|_{\mathcal{V}} := \sqrt{\langle x, x \rangle_{\mathcal{V}}}$.
3. Distance (of x and y): $d(x, y) := \|x \vec{-} y\|_{\mathcal{V}}$.

In the linear case, we shall construct projections from $\mathbb{R}^n$ to $\mathbb{R}^m$, $\forall m, n \in \mathbb{N}$.

Topology on DFES

Denote the topology induced by the above metric on $\mathbb{R}^\infty$ by $\mathcal{T}_{\rm d}$. The following result is crucial.

Proposition 2.1

The map $\mathrm{id} : (\mathbb{R}^\infty, \mathcal{T}_{\rm n}) \to (\mathbb{R}^\infty, \mathcal{T}_{\rm d})$ is continuous.

In other words, $\mathcal{T}_{\rm d} \subset \mathcal{T}_{\rm n}$.

Using d, we define the following equivalence relation on $\mathbb{R}^\infty$:

$$\forall x, y \in \mathbb{R}^\infty, \quad x \sim y \Leftrightarrow d(x, y) = 0.$$

Define $\Omega := \mathbb{R}^\infty / \sim$, equip Ω with the quotient topology of $\mathcal{T}_{\rm d}$, denoted by $\mathcal{T}$.

Hence completes the algebraic and topological constructions on $\mathbb{R}^\infty$:

Theorem 2.1

$\forall \bar{x}, \bar{y} \in \Omega$, $r \in \mathbb{R}$, define the addition $\vec{+}$ and scalar product as

$$\bar{x}\vec{+}\bar{y} := \overline{x\vec{+}y}, \qquad r \cdot \bar{x} := \overline{r \cdot x},$$

then $(\Omega, \mathcal{T})$ is a topological vector space under $\vec{+}$ and scalar product.

Further, $(\Omega, \mathcal{T})$ is a **pathwise connected Hausdorff** space, and we have the following diagram of morphisms:

$$\begin{array}{ccc} (\mathbb{R}^\infty, \mathcal{T}_{\mathrm{n}}) & \xrightarrow{\mathrm{id}} & (\mathbb{R}^\infty, \mathcal{T}_{\mathrm{d}}) \\ & \searrow^{q} & \downarrow \pi \\ & & (\Omega, \mathcal{T}) \end{array}$$

Structure of an Equivalence Class

One can see that the equivalence defined via the distance can also be rewritten as follows.

Definition 2.3

Let $x,\ y \in \mathcal{V}$. call $x,\ y$ as equivalent, denoted by $x \leftrightarrow y$, if there exists $\mathbf{1}_\alpha$, $\mathbf{1}_\beta$ such that

$$x \otimes \mathbf{1}_\alpha = y \otimes \mathbf{1}_\beta. \tag{3}$$

define the dimension of an element $\bar{x} \in \Omega$ as the smallest Euclidean dimension of the elements equivalent to x.

Covering Spaces

Definition 2.4

Let E, B be topological spaces, and $\Pi : E \to B$ is a continuous surjection. If for any open set $U \subset B$, we have $\pi^{-1}(U) = \bigsqcup_{\alpha} V_{\alpha}$, where V_{α} is open set in E, and $\forall \alpha$, $V_{\alpha} \simeq U$, then (E, π, B) is called a covering space.

Definition 2.5

Let (E_1, π_1, B_1), (E_2, π_2, B_2) be two covering spaces. If there exists homeomorphisms $\psi : E_1 \to E_2$, $\varphi : B_1 \to B_2$, such that $\pi_2 \circ \psi = \varphi \circ \pi_1$, then these two covering spaces are called homeomorphic.

Dimension-Free Manifolds

We define the dimension-free manifolds via local isomorphism with dimension-free Euclidean spaces.

Definition 2.6

Let B be a Hausdorff space with countable topological basis. A covering space $E \stackrel{P}{\to} B$ is called a dimension-free smooth Euclidean manifold, if it satisfies the following:

1. There exists a collection of open subsets $\{U_\alpha\}_{\alpha\in I} \subset B$, such that $\bigcup_{\alpha\in I} U_\alpha = B$, and $\forall \alpha \in I$, there exists an open set $V_\alpha \subset \Omega$ and homeomorphism $\varphi_\alpha : U_\alpha \to V_\alpha$, $\psi_\alpha : \pi^{-1}(U_\alpha) \to P^{-1}(V_\alpha)$, such that $\pi^{-1}(U_\alpha) \xrightarrow{\pi} U_\alpha$ and $P^{-1}(V_\alpha) \xrightarrow{P} V_\alpha$ are homeomorphisms of covering spaces.
2. $\forall \alpha, \beta \in I$, if $U_\alpha \cap U_\beta \neq \varnothing$, then $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ and $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha \circ \pi^{-1}(U_\alpha \cap U_\beta) \to \psi_\beta \circ \pi^{-1}(U_\alpha \cap U_\beta)$ are diffeomorphisms.

Construction of DFEMs

The compatibility of coordinate charts implies that the following diagram commutes:

$$\begin{array}{ccccc}
P^{-1}(V_\alpha \cap V_\beta) & \xleftarrow{\psi_\beta} & \pi^{-1}(U_\alpha \cap U_\beta) & \xrightarrow{\psi_\alpha} & P^{-1}(V_\alpha \cap V_\beta) \\
\downarrow P & & \pi \downarrow & & \downarrow P \\
V_\alpha \cap V_\beta & \xleftarrow{\varphi_\beta} & U_\alpha \cap U_\beta & \xrightarrow{\varphi_\alpha} & V_\alpha \cap V_\beta
\end{array}$$

Projections Between Euclidean Spaces

Definition 2.7

The projection from $\mathbb{R}^n$ to $\mathbb{R}^m$, denoted by Π_m^n, is defined as

$$\Pi_m^n(\xi) := \underset{x\in\mathbb{R}^m}{\operatorname{argmin}}\ d(\xi, x), \quad \forall \xi \in \mathbb{R}^n \tag{4}$$

where the distance is defined as in Definition 1.1.

Proposition 2.2

$\forall m, n \in \mathbb{N}$, assume $\operatorname{lcm}(n, m) = t$, $\alpha := t/n$, $\beta := t/m$.
$\Pi_m^n : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator, with a matrix representation as

$$\Pi_m^n = \frac{1}{\beta}\left(I_m \otimes \mathbf{1}_\beta^T\right)\left(I_n \otimes \mathbf{1}_\alpha\right). \tag{5}$$

Moreover, $\langle \Pi_m^n(\xi), \xi - \Pi_m^n(\xi)\rangle_{\mathcal{V}} = 0$.

Functions Over Dimension-Free Manifolds

Definition 2.8

Let $f : \Omega \to \mathbb{R}$ be a real function on Ω. Define $\tilde{f} : \mathbb{R}^\infty \to \mathbb{R}$, $x \mapsto f(\bar{x})$. If $\tilde{f} \in C^\infty(\mathbb{R}^\infty)$ (with respect to the Euclidean coordinates), then f is called a smooth function on Ω.

Proposition 2.3

Let $f \in C^r(\mathbb{R}^n)$. Define $\bar{f} : \Omega \to \mathbb{R}$ as follows: Let $\bar{x} \in \Omega$ and $\dim(\bar{x}) = m$. Then

$$\bar{f}(\bar{x}) := f(\Pi_n^m(x_1)), \quad \bar{x} \in \Omega, \tag{6}$$

is smooth on Ω, where $x_1 \in \bar{x}$ is the smallest element in $\bar{x}$.

Tangent Space of Dimension-Free Manifolds

Definition 2.9

Let $M = (E, P, B)$ be a smooth dimension-free Euclidean manifold. Denote the germ of smooth functions at $p \in B$ as $C_p^\infty(B)$. Consider a map $v : C_p^\infty(B) \to \mathbb{R}$, if $\forall [f], [g] \in C_p^\infty(B)$,

1. (Linearity) $\forall \alpha, \beta \in \mathbb{R}$, $v(\alpha[f] + \beta[g]) = \alpha v([f]) + \beta v([g])$;
2. (Leibniz rule) $v([f][g]) = f(p)v([g]) + g(p)v([f])$,

then v is called a tangent vector over M. The set of all tangent vectors are called the tangent space of (E, P, B) at p.

Properties of Tangent Space

Proposition 2.4

$\forall v \in T_p(E,P,B)$, $\exists \bar{v} : P_*(C_p^\infty(B)) \to \mathbb{R}$, such that $v = \bar{v} \circ P_*$, where $P_* : C_p^\infty(B) \to C_p^\infty(E)$, $f \mapsto f \circ P$.

Proposition 2.5

The tangent space $T_p(E,P,B)$ is a linear space homomorphic to $\mathbb{R}^{\dim(p)}$.

Vector Fields Over Dimension-Free Manifolds

Definition 2.10

$\bar{X}$ is called a smooth vector field on Ω, denoted by $\bar{X} \in \mathfrak{X}(\Omega)$, if it satisfies the following conditions:

(i) At each point $\bar{x} \in \Omega$, there exists $p = \mu \dim(\bar{x})$, called the dimension of the vector field $\bar{X}$ at $\bar{x}$ and denoted by $\dim(\bar{X}_{\bar{x}})$, such that $\bar{X}$ assigns a p sub-lattice to the bundle of coordinate neighborhood at $\bar{x}$, $\mathcal{V}_O^{[p,\cdot]} = \{O^p, O^{2p}, \cdots\}$, then at each leaf of this sub-lattice the vector field assigns a vector $X^j \in T_{x_{j\mu}}(O^{jp})$, $j = 1, 2, \cdots$.

(ii) $\{X^j \mid j = 1, 2, \cdots\}$ satisfy consistence condition, that is, $X^j|_{x_j \otimes \mathbf{1}_j} = X^1 \otimes \mathbf{1}_j$, $x_j \in O^{jp}$, $j = 1, 2, \cdots$.

(iii) At each leaf $O^{jp} \subset \mathbb{R}^{j\mu \dim(\bar{x})}$,

$$\bar{X}|_{O^{jp}} \in \mathfrak{X}(O^{jp}). \tag{7}$$

Construction of Vector Fields

Algorithm 2.1

- Step 1: Assume there exists a smallest dimension $m > 0$, such that $\bar{X}$ is defined over whole $\mathbb{R}^m$. That is,

$$\bar{X}|_{\mathbb{R}^m} := X \in \mathfrak{X}(\mathbb{R}^m). \tag{8}$$

  From the constructing point of view: A vector field $X \in \mathfrak{X}(\mathbb{R}^m)$ is firstly given, such that the value of $\bar{X}$ at leaf $\mathbb{R}^m$ is uniquely determined by (8).

- Step 2: Extend X to $T_{\bar{y}}$. Assume $\dim(\bar{y}) = s$, denote $m \vee s = t$. Then $\dim(T_{\bar{y}}) = t$. Let $y \in \bar{y} \bigcap R^{[t,\cdot]}$, and $\dim(y) = kt$, $k = 1, 2, \cdots$. Define

$$\bar{X}(y) := \Pi^m_{kt} X(\Pi^{kt}_m y), \quad k = 1, 2, \cdots. \tag{9}$$

An Example

Let $X=(x_1+x_2, x_2^2)^T \in \mathfrak{X}(\mathbb{R}^2)$. Assume $\bar{X} \in \mathfrak{X}(\Omega)$ is generated by X. Consider $\bar{y} \in \Omega$, $\dim(\bar{y})=3$, Denote $y_1=(\xi_1,\xi_2,\xi_3)^T \in \mathbb{R}^3$. Since $2 \vee 3 = 6$, $\bar{X}$ at $\bar{y} \bigcap \mathbb{R}^{6k} = \{y_2, y_4, y_6, \cdots\}$ is well defined.
Now consider y_2.

$$
\begin{aligned}
\bar{X}(y_2) = \Pi_6^2 X(\Pi_2^6(y_2)) = & \quad (I_2 \otimes \mathbf{1}_3) X\left(\tfrac{1}{3}(I_2 \otimes \mathbf{1}_3^T)(y_1 \otimes \mathbf{1}_2)\right) \\
= & \quad \begin{bmatrix} \frac{2}{3}(\xi_1+\xi_2+\xi_3) \\ \frac{2}{3}(\xi_1+\xi_2+\xi_3) \\ \frac{2}{3}(\xi_1+\xi_2+\xi_3) \\ \frac{1}{9}(\xi_2+2\xi_3)^2 \\ \frac{1}{9}(\xi_2+2\xi_3)^2 \\ \frac{1}{9}(\xi_2+2\xi_3)^2 \end{bmatrix}.
\end{aligned}
$$

Consider y_4, similar calculation shows that

$$
\bar{X}(y_4) = \Pi_{12}^2 X(\Pi_2^{12}(y_4)) = \bar{X}(y_2) \otimes \mathbf{1}_2.
$$

In fact, we have

$$
\bar{X}(y_{2k}) = \bar{X}(y_2) \otimes \mathbf{1}_k, \quad k = 1, 2, \cdots .
$$

An Example (Cont'd)

Consider $\bar{X}|_{\mathbb{R}^6}$:
Assume $z = (z_1, z_2, z_3, z_4, z_5, z_6)^T \in \mathbb{R}^6$. Then

$$X^6 := \bar{X}_z = \Pi_6^2 X(\Pi_2^6 z) = \begin{bmatrix} \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ \frac{1}{3}(z_1 + z_2 + z_3 + z_4 + z_5 + z_6) \\ \frac{1}{9}(z_4 + z_5 + z_6)^2 \\ \frac{1}{9}(z_4 + z_5 + z_6)^2 \\ \frac{1}{9}(z_4 + z_5 + z_6)^2 \end{bmatrix}. \tag{10}$$

$X^6 \in \mathfrak{X}(\mathbb{R}^6)$ is a standard vector field.

Generalizing DFES to Manifolds

We begin with constructing dimension-free manifolds as quotient spaces.

Definition 2.11

Given a Riemannian manifold M and an isometry $\varphi : M \to M$ over it, denote by $M^n := \underbrace{M \times \cdots \times M}_{n}$ the n-fold Cartesian product of M, endowed with the product topology.

The dimension-free manifold generated by (M, φ) is defined as

$$\tilde{M} := M^\infty / \sim \qquad (11)$$

where $M^\infty := \bigsqcup_{n=1}^{\infty} M^n$, and the equivalence relation over M^∞ is defined as

$$\forall s > 0,\ \forall k > 0,\ \forall (x_1, \cdots, x_s) \in M^s,$$
$$(x_1, \cdots, x_s) \sim (x_1, \cdots, x_s, \varphi(x_1), \cdots, \varphi(x_s), \cdots, \varphi^k(x_1), \cdots, \varphi^k(x_s)).$$

Vector Fields Over Dimension-Free Manifolds

Proposition 2.6

A vector field $f \in \mathfrak{X}(M^s)$ can be extended to $\tilde{M}$ in the following way: assume $y \sim (x_1, \cdots, x_s)$, without loss of generality, let $y = (x_1, \cdots, x_s, \varphi(x_1), \cdots, \varphi(x_s), \cdots, \varphi^k(x_1), \cdots, \varphi^k(x_s))$, then the value of extended vector field $\tilde{f}$ at y can be defined as:

$$\begin{aligned} \tilde{f}|_y \quad := \quad & (f_1|_{x_1}, \cdots, f_s|_{x_s}, \varphi_*(f_1|_{x_1}), \cdots, \varphi_*(f_s|_{x_s}), \cdots, \\ & \varphi_*^k(f_1|_{x_1}), \cdots, \varphi_*^k(f_s|_{x_s})). \end{aligned}$$

One can easily see that if we take the manifold M in Definition 12 as $\mathbb{R}$ and let $\varphi := \mathrm{id}$, then the above definition coincides with our general form. Similarly, in the linear case, $\tilde{f}|_{x \otimes \mathbf{1}_k} := f|_x \otimes \mathbf{1}_k$.

Dimension-Varying Linear Systems

Consider a linear system over $\mathbb{R}^n$ as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{12}$$

where $u \in \mathbb{R}^m$. Using the projection we can construct a least square approximate system of (12) on $\mathbb{R}^m$, $\forall m \in \mathbb{N}$.

$$\Sigma_m : \ \dot{\xi}(t) = A_{\Pi}\xi(t) + \Pi_m^p Bu(t)$$

where

$$A_{\Pi} = \begin{cases} \Pi_m^n A(\Pi_m^n)^{\mathrm{T}} \left(\Pi_m^n(\Pi_m^n)^{\mathrm{T}}\right)^{-1} & n \geqslant m \\ \Pi_m^n A \left((\Pi_m^n)^{\mathrm{T}} \Pi_m^n\right)^{-1} (\Pi_m^n)^{\mathrm{T}} & n < m. \end{cases} \tag{13}$$

Hence we have defined a family of linear systems on $\mathbb{R}^\infty$ corresponding to the system Σ.

Projection Systems

Definition 3.1

Let

$$\dot{\bar{x}} = \bar{F}(\bar{x}), \quad \bar{x} \in \Omega. \tag{14}$$

be a system on Ω. A dynamic system

$$\dot{x} = F(x), \quad x \in \mathbb{R}^n \subset \mathbb{R}^\infty, \tag{15}$$

is called a realization (or a lifting) of (14), if for each $\bar{x}$ there exists $x \in \bar{x}$, such that the corresponding vector field $F(x) \in \bar{F}(\bar{x})$. Meanwhile, system (14) is called the projection system of (15).

Lift and Projection of Control Systems

Definition 3.2

Consider control system

$$\Sigma^C: \ \dot{x} = F(x, u), \quad x \in \mathbb{R}^p, \ u \in \mathbb{R}^r. \tag{16}$$

$u = u_1, \cdots, u_r$ can be considered as controlled parameters. Then its projection to R^q is

$$\mathit{\Pi}_q^p(\Sigma^C): \ \dot{z} = \tilde{F}(z, u), \quad z \in \mathbb{R}^q, \ u \in \mathbb{R}^r, \tag{17}$$

where

$$\tilde{F}(z, u) = \mathit{\Pi}_q^p F(\mathit{\Pi}_p^q(z), u). \tag{18}$$

Integral Curves

Definition 3.3

Let $\bar{X} \in \mathfrak{X}(\Omega)$. $\bar{x}(t, \bar{x}_0)$ is called the integral curve of $\bar{X}$ with initial value $\bar{x}_0$, denoted by $\bar{x}(t, \bar{x}_0) = \Phi_t^{\bar{X}}(\bar{x}_0)$, if for each initial value $x_0 \in \bar{x}_0 \bigcap \mathbb{R}^n$, and each generator of $\bar{X}$, denoted by $X = \bar{X}|_{\mathbb{R}^n}$, the following condition holds:

$$\Phi_t^{\bar{X}}(\bar{x}_0)|_{\mathbb{R}^n} = \overline{\Phi_t^{X}(x_0)}, \quad t \geq 0. \tag{19}$$

Integral Curves of Linear Vector Fields

Proposition 3.1

Let $\bar{X} \in \mathfrak{X}(\Omega)$ be a linear vector field, and $\dim(\bar{X}) = m$. $X := \bar{X}|_{\mathbb{R}^m} = Ax$. Assume $\bar{x}^0 \in \Omega$, $\dim(\bar{x}^0) = s$.

(i) If $s = m$, then the integral curve of $\bar{X}|_{\mathbb{R}^m}$ is $\Phi_t^X(x_1^0) = \mathrm{e}^{Xt}x_1^0$. Hence, the integral curve of $\bar{X}|_{\mathbb{R}^{rm}}$ becomes $\Phi_t^{X_r}(x_r^0) = \left[\mathrm{e}^{Xt}x_1^0\right] \otimes \mathbf{1}_r$. The integral curve of $\bar{X}$ with initial value $\bar{x}^0$ is $\overline{\Phi_t^X(x_1^0)} \subset \Omega$.

(ii) If $s = km$, then the integral curve of $\bar{X}|_{\mathbb{R}^{km}}$ is $\Phi_t^{X_k}(x_1^0) = \mathrm{e}^{X_k t}x_1^0$, where

$$X_k := \bar{X}(y) = \Pi_{km}^m(X\left(\Pi_m^{km}(y)\right)) = \Pi_{km}^m A \Pi_m^{km} y := A_k y, \ \ y \in \mathbb{R}^{km}.$$

. Hence the integral curve of $\bar{X}$ with initial value $\bar{x}^0$ is $\overline{\Phi_t^{X_k}(x_1^0)} \subset \Omega$.

(iii) If $m \vee s = p = km = rs$, then the integral curve of $\bar{X}|_{\mathbb{R}^p}$ is $\Phi_t^{X_k}(x_r^0) = \mathrm{e}^{X_k t}(x_1^0 \otimes I_s)$. Hence, the integral curve of $\bar{X}$ with initial value $\bar{x}^0$ is $\overline{\Phi_t^{X_k}(x_1^0 \otimes I_s)} \subset \Omega$.

Definition 3.4

Consider an affine nonlinear control system on Ω, described by

$$\begin{cases} \dot{\bar{x}}(t) = \bar{f}(x) \vec{+} \sum_{i=1}^{m} \bar{g}_i(x) u^i, \\ \bar{y}_j(t) = \bar{h}_j(\bar{x}(t)), \quad j \in [1,p], \end{cases} \tag{20}$$

where $\dim(\bar{f}) = n_0$, $\dim(\bar{g}_i) = n_i$, $i \in [1,m]$, $\dim(\bar{h}_j) = r_j$, $j \in [1,p]$. Let $n = \left(\bigvee_{i=0}^{m} n_i\right) \bigvee \left(\bigvee_{j=1}^{p} r_j\right)$, then

$$\begin{cases} \dot{\bar{x}}(t) = f^n(x) \vec{+} \sum_{i=1}^{m} g_i^n(x) u_i, \\ y_j(t) = h_j^n(\bar{x}(t)), \quad j \in [1,p], \end{cases} \tag{21}$$

where $f^n = \bar{f}|_{\mathbb{R}^n}$, $g_i^n = \bar{g}_i|_{\mathbb{R}^n}$, $i \in [1,m]$, $h_j^n = \bar{g}_j|_{\mathbb{R}^n}$, $j \in [1,p]$. (21) is called the minimum realization of (20).

An Example

Consider a linear control system $\bar{\Sigma}$ over Ω, which has its dynamic equation as Eq. (21), where $\bar{f}$ has its smallest generator $f(x) = 2[x_1 + x_2, x_2]^{\mathrm{T}} \in \mathfrak{X}(\mathbb{R}^2)$, $m = 2$, the smallest generator of $\bar{g}_1$ is $g_1 = [1, 0, 0, 1]^{\mathrm{T}} \in \mathfrak{X}(\mathbb{R}^4)$, the smallest generator of $\bar{g}_2$ is $g_2 = [0, 1, 0, 0]^{\mathrm{T}} \in \mathfrak{X}(\mathbb{R}^4)$. $p = 1$, $\bar{h}|_{\mathbb{R}^2} = x_2 - x_1$. Then, $q = 4$. We try to analyse the control properties concerning this system.

An Example (Cont'd)

$$\bar{f}|_{\mathbb{R}^4} = \Pi_4^2 f\left(\Pi_2^4[z_1, z_2, z_3, z_4]^{\mathrm{T}}\right) = \begin{bmatrix} z_1 + z_2 + z_3 + z_4 \\ z_1 + z_2 + z_3 + z_4 \\ z_3 + z_4 \\ z_3 + z_4 \end{bmatrix} := Az,$$

where,

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

$$\bar{h}|_{\mathbb{R}^4} = h(\Pi_2^4 z) = h(z_1 + z_2, z_3 + z_4) = z_1 + z_2 - z_3 - z_4 := Cz,$$

where $C = [1, 1, -1, -1]$.

An Example (Cont'd)

Then the smallest generator of system $\bar{\Sigma}$, denoted by $\Sigma := \bar{\Sigma}|_{\mathbb{R}^4}$, is

$$\begin{cases} \dot{z} = Az + Bu, \\ y = Cz. \end{cases}$$

Then it is easy to calculate that the controllability matrix of Σ is

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 2 & 1 & 6 & 2 & 16 & 4 \\ 0 & 1 & 2 & 1 & 6 & 2 & 16 & 4 \\ 0 & 0 & 1 & 0 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 & 4 & 0 \end{bmatrix}.$$

Since $\mathrm{rank}(\mathcal{C}) = 4$, Σ is completely controllable. By definition, $\bar{\Sigma}$ is completely controllable.

Conclusions and Outlook

Possible applications for DFEMs:

- Dimension-varying systems;
- Projection of high-dimensional systems;
- Synchronous multi-agent systems.

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Thanks for your attention!