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Stability Analysis and Feedback Stabilization of Probabilistic Logic Dynamical Systems

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Outline

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Notations

- $\mathscr{D}_n$: n-valued logic domain $\mathscr{D}_n = \{1, 2, \cdots, n\}$
- Δ_n: vector-form of logic domain $\mathscr{D}_n$, $\Delta_n = \mathrm{Col}(I_n)$
- δ_n^j: vector-form of $j \in \mathscr{D}_n$, $\delta_n^j = \mathrm{Col}_j(I_n)$
- $\vec{x}$: vector-form of logic variable $x \in \mathscr{D}_n$
- $\mathbf{R}_{[n]}$: power-reducing matrix

- A **logic dynamical system (LDS)** is a dynamical system evolves within the logic domain $\mathscr{D}_n := \{1, 2, \cdots, n\}$.

$$x_{t+1} = f(x_t)$$

  - $x_t \in \mathscr{D}_n$, $f : \mathscr{D}_n \to \mathscr{D}_n$

[1] Stuart A Kauffman. "Metabolic stability and epigenesis in randomly constructed genetic nets". In: *Journal of Theoretical Biology* 22.3 (1969), pp. 437–467.

[2] Gautier Stoll et al. "Continuous time boolean modeling for biological signaling: application of Gillespie algorithm". In: *Bmc Systems Biology* 6.1 (2012), pp. 116–116.

1.1 Probabilistic Logic Dynamical Systems

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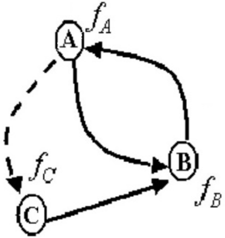
- **A Typical Example - Boolean network:** A special LDS proposed by Kauffman[1] as a qualitative model for GRNs.
  - Even though a BN provides a rougher description of GRNs, it is still capable of efficiently predicting the long-term behavior of GRNs[2].

[1] Stuart A Kauffman. "Metabolic stability and epigenesis in randomly constructed genetic nets". In: *Journal of Theoretical Biology* 22.3 (1969), pp. 437–467.

[2] Gautier Stoll et al. "Continuous time boolean modeling for biological signaling: application of Gillespie algorithm". In: *Bmc Systems Biology* 6.1 (2012), pp. 116–116.

- **An Example Boolean Network**



$f_A(B) = B$

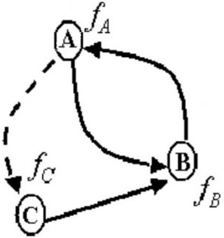
$f_B(A, C) = A \wedge C$

$f_C(A) = \neg A$

Regulatory functions

1.1 Probabilistic Logic Dynamical Systems

- **An Example Boolean Network**



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Regulatory functions

$$\begin{cases} A_{t+1} = B_t \\ B_{t+1} = A_t \wedge C_t \\ C_{t+1} = \neg A_t \end{cases}$$

Dynamical equation

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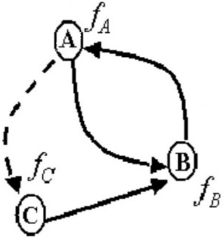

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Regulatory functions

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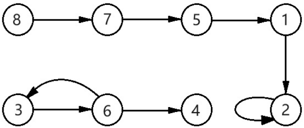
Dynamical equation

State	A_t	B_t	C_t	A_{t+1}	B_{t+1}	C_{t+1}
1	0	0	0	0	0	1
2	0	0	1	0	0	1
3	0	1	0	1	0	1
4	0	1	1	1	0	1
5	1	0	0	0	0	0
6	1	0	1	0	1	0
7	1	1	0	1	0	0
8	1	1	1	1	1	0

Truth table



State transition graph

1.1 Probabilistic Logic Dynamical Systems

- A **probabilistic logic dynamical system (PLDS)** is a collection of LDSs driven by a random process

$$x_{t+1} = f(w_t, x_t)$$

  - $w_t \in \mathscr{D}_{n_w}$ is the **random disturbance** (i.i.d. process, Markov chain, or state-dependent process)
  - $f : \mathscr{D}_{n_w} \times \mathscr{D}_n \to \mathscr{D}_n$

[3] Ilya Shmulevich, Edward R Dougherty, and Wei Zhang. "From Boolean to probabilistic Boolean networks as models of genetic regulatory networks". In: *Proceedings of the IEEE* 90.11 (2002), pp. 1778–1792.

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- **A Typical Example - Probabilistic Boolean Network (PBN):** A stochastic generalization of deterministic BN, aiming to describe uncertainties and stochasticity in GRNs[3].

[3] Ilya Shmulevich, Edward R Dougherty, and Wei Zhang. "From Boolean to probabilistic Boolean networks as models of genetic regulatory networks". In: *Proceedings of the IEEE* 90.11 (2002), pp. 1778–1792.

- A **PBN** is a randomly switched Boolean network

$$
\begin{cases}
x_1(t+1) = f_1^{\sigma_1(t)}\left(\left\{x_j(t) \mid j \in \mathcal{N}_1^{\sigma_1(t)}\right\}\right) \\
x_2(t+1) = f_2^{\sigma_2(t)}\left(\left\{x_j(t) \mid j \in \mathcal{N}_2^{\sigma_2(t)}\right\}\right) \\
\vdots \\
x_n(t+1) = f_n^{\sigma_n(t)}\left(\left\{x_j(t) \mid j \in \mathcal{N}_n^{\sigma_n(t)}\right\}\right)
\end{cases} \tag{1}
$$

  - $x_i \in \mathscr{B} := \{0,1\} \sim \mathscr{D}_2$;
  - $\sigma_i(t) \in \mathscr{D}_{N_i}$, $i = 1, 2, \cdots, n$, are random switching sequences; and
  - f_i^j, $i \in [1:n]$, $j \in \mathscr{D}_{N_i}$, are Boolean functions of their respective in-neighbouring nodes $\left\{x_k(t) \mid k \in \mathcal{N}_i^j\right\}$.
  - There are $N := \Pi_{i=1}^n N_i$ subnetworks in total.

- **Algebraic Form of PLDS**

$$x_{t+1} = f(w_t, x_t)$$

$$\Updownarrow$$

$$\vec{x}_{t+1} = L_f \ltimes \vec{w}_t \ltimes \vec{x}_t$$

  - $\vec{x}_t := \delta_n^{x_t}$ and $\vec{w}_t := \delta_{n_w}^{w_t}$ are the vector-forms of x_t and w_t, respectively.
  - $L_f \in \mathscr{L}_{n \times n n_w}$ is the structural matrix of logic function f, obtained from its truth table:

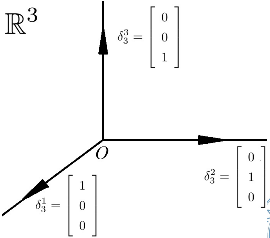
$$\mathrm{Col}_{(w-1)n+j}(L_f) = \vec{f}(w, j) = \delta_n^{f(w,j)}, \quad w \in \mathscr{D}_{n_w}, j \in \mathscr{D}_n.$$

- **Why Using Algebraic Form?**

  The STP and the vector-representation of logic

  - transform the logical calculations into algebraic calculations, and
  - embed a LDS into the Euclidean space $\mathbb{R}^n$, enabling us to study LDSs using the structure of Euclidean space.

$$\vec{x}_{t+1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}_t, \quad x_0 \in \Delta_3$$



1.1 Probabilistic Logic Dynamical Systems

- **I.i.d. Switching Case** (Most studied case in literature)
  - **Basic assumptions:**
    - ⋆ w_t is an i.i.d. random sequence

$$w_t \sim \boldsymbol{p}^w, \quad [\boldsymbol{p}^w]_j := \mathbb{P}\{w_t = j\}.$$

    - ⋆ For any t, w_t is independent of state history $\{x_s \mid s \leq t\}$.
  - **Markovian Property:** x_t is a homogeneous Markov chain
    - ⋆ **Transition probability matrix (TPM):**

$$\mathbf{P} = L_f \ltimes \boldsymbol{p}^w$$

$$[\mathbf{P}]_{i,j} = \mathbb{P}\{x_{t+1} = i \mid x_t = j\}, \quad i, j \in \mathscr{D}_n$$

Note: Conventionally, the TPM is defined as $\mathbf{P}^\top$.

    - ⋆ **Dynamics of State PDV $\boldsymbol{\pi}_t$:** $x_t \sim \boldsymbol{\pi}_t := \mathbb{E}\vec{x}_t \in \Upsilon_n$

$$\boldsymbol{\pi}_{t+1} = \mathbf{P}\boldsymbol{\pi}_t$$

1.1 Probabilistic Logic Dynamical Systems

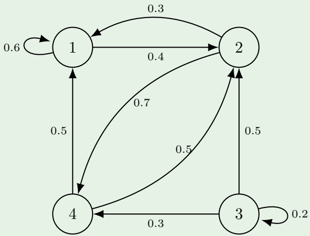
- **State Transfer Graph (STG):**

  The STG of a PLS is a weighted directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, W)$ where

  - $\mathcal{N} = \mathscr{D}_n$ or Δ_n is the set of nodes;
  - $\mathcal{E} = \{(j,i) \mid [\mathbf{P}]_{i,j} > 0\}$ is the set of directed edges;
  - $W : \mathcal{E} \to (0,1]$, $(j,i) \mapsto [\mathbf{P}]_{i,j}$, is the weight of edge.

Example 1

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.3 & 0 & 0.5 \\ 0.4 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.7 & 0.3 & 0 \end{bmatrix}$$



Lemma 2

For any $i, j \in \mathscr{D}_n$, the following statements are equivalent:

- $[\mathbf{P}^t]_{j,i} > 0$ *for some* t *with* $1 \leq t \leq n-1$;
- *The STG* $(\mathcal{N}, \mathcal{E}, W)$ *has a path from* i *to* j*, denoted by* $i \to j$.

1.1 Probabilistic Logic Dynamical Systems

- **Stationary Distribution and Its Convergence**
  - **Stationary distribution:** A distribution $\boldsymbol{\pi} \in \Upsilon_n$ satisfying $\mathbf{P}\boldsymbol{\pi} = \boldsymbol{\pi}$.
    - ★ If $\boldsymbol{\pi}$ is a stationary distribution, then, $x_0 \sim \boldsymbol{\pi}$ implies $x_t \sim \boldsymbol{\pi}\ \forall t$
    - ★ A **Finite** Markov chain (Thus, a PLDS) has at least one stationary distribution.
  - **Basic Limit Theorem:** Let x_t be an irreducible, aperiodic Markov chain having a stationary distribution $\boldsymbol{\pi}$. Then

$$\lim_{t\to\infty} \boldsymbol{\pi}_t = \lim_{t\to\infty} \mathbf{P}^t \boldsymbol{\pi}_0 = \boldsymbol{\pi} \quad \forall \boldsymbol{\pi}_0 \in \Upsilon_n.$$

Note: Please notice the difference between the convergence of stationary distribution and the (set) stability discussed later.

- **Fixed Point and Invariant Set (Closed Set)**
  - A subset $\mathcal{C} \subset \mathscr{D}_n$ is called an **invariant subset** if
  $$\mathbb{P}\left\{x_{t+1} \in \mathcal{C} \;\middle|\; x_t \in \mathcal{C}\right\} = 1.$$
  - A state x_e is called a **fixed point** if $\{x_e\}$ is invariant.

- **Fixed Point and Invariant Set (Closed Set)**
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Lemma 3

The transition probability from any state to an invariant subset $\mathcal{C}$ is non-decreasing with time, that is, for any $k \in \mathbb{Z}_+$ and any $j \in \mathscr{D}_n$,

$$\mathbb{P}\{x_{t+k} \in \mathcal{C} \mid x_0 = j\} \geq \mathbb{P}\{x_t \in \mathcal{C} \mid x_0 = j\}$$

- **The Largest Invariant Subset**
  - The union of two invariant subsets is still invariant.
  - The union of all invariant subsets contained in $\mathcal{M}$ is referred to as the **largest invariant subset** in $\mathcal{M}$, denoted by $I(\mathcal{M})$.

Proposition 1

For a given subset $\mathcal{M} \subseteq \mathscr{D}_n$, we define a sequence of subsets as[a]

$$\mathcal{M}_s = \left\{ j \in \mathcal{M}_{s-1} \;\middle|\; \sum_{i \in \mathcal{M}_{s-1}} [\mathbf{P}]_{i,j} = 1 \right\}, \quad s = 1, 2, \cdots ,$$

where $\mathcal{M}_0 := \mathcal{M}$. Then, there must exist an integer $\mathbf{k} \leq |\mathcal{M}|$ such that $\mathcal{M}_{\mathbf{k}} = \mathcal{M}_{\mathbf{k}-1}$. In addition, it holds that $I(\mathcal{M}) = \mathcal{M}_{\mathbf{k}}$.

[a] Yuqian Guo et al. "Stability and Set Stability in Distribution of Probabilistic Boolean Networks". In: *IEEE Transactions on Automatic Control* 64 (2 2019), pp. 736–742.

1.1 Probabilistic Logic Dynamical Systems

- **Probabilistic Logic Dynamical Control Systems (PLDCS)**

$$\begin{cases} x_{t+1} = f(w_t, u_t, x_t) \\ y_t = h(v_t, x_t) \end{cases}$$

  - $x_t \in \mathscr{D}_n$, $u_t \in \mathscr{D}_m$, $y_t \in \mathscr{D}_q$
  - $f : \mathscr{D}_{n_w} \times \mathscr{D}_m \times \mathscr{D}_n \to \mathscr{D}_n; \quad h : \mathscr{D}_{n_v} \times \mathscr{D}_n \to \mathscr{D}_q$
  - $w_t \sim \boldsymbol{p}^w$

$$\Updownarrow$$

$$\begin{cases} \vec{x}_{t+1} = L_f \ltimes \vec{w}_t \ltimes \vec{u}_t \ltimes \vec{x}_t \\ \vec{y}_t = L_h \ltimes \vec{v}_t \ltimes \vec{x}_t \end{cases}$$

  - $L_f \in \mathscr{L}_{n \times n_w m n}$, $L_h \in \mathscr{L}_{q \times n_v n}$

- **Basic assumptions:**
  - w_t and v_t are i.i.d. random sequences that are mutually independent.

$$w_t \sim \boldsymbol{p}^w, \quad v_t \sim \boldsymbol{p}^v.$$

  - For any t, w_t and v_t are independent of state history $\{x_s \mid s \leq t\}$.
- **TPMs**

$$\mathbf{P} = L_f \ltimes \boldsymbol{p}^w$$

$$\mathbf{P}_j = L_f \ltimes \boldsymbol{p}^w \ltimes \delta_m^j$$

- **Reachability**
  - x_d is said to be k-step reachable from x_0 if there is a control sequence $\mathbf{u} = \{u(t)\}$ such that

$$\mathbb{P}\{x(k; x_0, \mathbf{u}) = x_d\} > 0.$$

    x_d is said to be reachable from x_0, denoted by $x_0 \xrightarrow{u} x_d$, if there is a control sequence $\mathbf{u} = \{u(t)\}$ such that

$$\mathbb{P}\{x(t; x_0, \mathbf{u}) = x_d \text{ for some } t \geq 1\} > 0.$$

  - x_d is reachable from x_0 if and only if x_d is k-step reachable from x_0 for some $k \leq 2^n - 1$.

- Reachability Matrix

$$\mathbf{R} = \sum_{k=1}^{n-1} (\mathbf{P} \ltimes \mathbf{1}_m)^k$$

Proposition 2

$i \xrightarrow{u} j$ *iff* $[\mathbf{R}]_{j,i} > 0$.

Sketchy Proof:

$$\begin{aligned} (\mathbf{P} \ltimes \mathbf{1}_m)^k &= (\mathbf{P}_1 + \mathbf{P}_2 + \cdots + \mathbf{P}_m)^k \\ &= \sum_{\text{all possible combinations}} \mathbf{P}_{i_k} \cdots \mathbf{P}_{i_2} \mathbf{P}_{i_1} \end{aligned}$$

Thus, $\left[(\mathbf{P} \ltimes \mathbf{1}_m)^k \right]_{j,i} > 0$ if and only if j is k-step reachable from i.

1.1 Probabilistic Logic Dynamical Systems

- **Control Invariant Subsets**
  - A subset $\mathcal{C} \subseteq \mathscr{D}_n$ is termed as a control invariant subset if, for any state $j \in \mathcal{C}$, there exists a control $r \in \mathscr{D}_m$ such that

$$\mathbb{P}\{x_{t+1} \in \mathcal{C} \mid x_t = j, u_t = r\} = 1. \tag{2}$$

  - The union of any two control invariant subsets is still control invariant.
  - The union of all control invariant subsets contained in a given subset $\mathcal{M} \subseteq \mathscr{D}_n$ is termed as the **largest control invariant subset** contained in $\mathcal{M}$ and is denoted by $I_c(\mathcal{M})$.
  - If $\mathcal{C} = \{x_e\}$ is control invariant, then, x_e is called a control fixed point.

Proposition 3

Suppose that $\mathcal{M}_0 \subseteq \mathscr{D}_n$. *A sequence of subsets* $\mathcal{M}_s, s \in \mathbb{Z}^+$, *is defined as*

$$\mathcal{M}_s = \left\{ j \in \mathcal{M}_{s-1} \middle| \exists k \in [1:m], \text{s.t.} \sum_{i \in \mathcal{M}_{s-1}} [\mathbf{P}_k]_{i,j} = 1 \right\}.$$

Then, there must exist a positive integer $\eta \leqslant |\mathcal{M}_0|$ *such that* $\mathcal{M}_\eta = \mathcal{M}_{\eta+1}$. *Additionally,* $I_c(\mathcal{M}_0) = \mathcal{M}_\eta$ *holds.*

Outline

1.2 Nonnegative Matrices

- **Nonnegative Matrices:** A matrix A is called a nonnegative matrix, denoted as $A \succeq 0$, if it is nonnegative element-wise, that is, all of its elements are nonnegative.

Definition 4

Consider two $m \times q$ nonnegative matrices $\Gamma_1 \succeq 0$ and $\Gamma_2 \succeq 0$.

- Γ_1 is said to be structurally included in Γ_2, denoted as $\Gamma_1 \sqsubseteq \Gamma_2$, if for any $i \in [1:m]$ and any $j \in [1:q]$, $[\Gamma_2]_{i,j} = 0$ implies $[\Gamma_1]_{i,j} = 0$.
- They are said to be homo-structural, denoted as $\Gamma_1 \sim_h \Gamma_2$, if both $\Gamma_1 \sqsubseteq \Gamma_2$ and $\Gamma_2 \sqsubseteq \Gamma_1$ hold.

Lemma 5

Consider $m \times n$ nonnegative matrices $A, B \succeq 0$ and $p \times q$ nonnegative matrices $C, D \succeq 0$. If $A \sqsubseteq B$ and $C \sqsubseteq D$, then it holds that

$$A \ltimes C \sqsubseteq B \ltimes D.$$

Outline

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- Consider PLDS

$$x_{t+1} = f(w_t, x_t)$$

  - $x_t \in \mathscr{D}_n$, $w_t \sim \boldsymbol{p}^w \in \Upsilon_{n_w}$
  - $f : \mathscr{D}_{n_w} \times \mathscr{D}_n \to \mathscr{D}_n$

Definition 6 (Finite-time Stability(FTS))

A state $x_e \in \mathscr{D}_n$ is said to be finite-time stable if there is a positive integer T such that[a]

$$\mathbb{P}\{x_t = x_e \mid x_0 = j\} = 1 \quad \forall t \geq T, \forall j \in \mathscr{D}_n.$$

[a] Rui Li, Meng Yang, and Tianguang Chu. "State feedback stabilization for probabilistic Boolean networks". In: *Automatica* 50.4 (2014), pp. 1272–1278.

Definition 7 (Stability with Probability One (SPO))

A state $x_e \in \mathscr{D}_n$ is said to be stable with probability one if[a]

$$\mathbb{P}\left\{\lim_{t\to\infty} x_t = x_e \;\middle|\; x_0 = j\right\} = 1 \quad \forall j \in \mathscr{D}_n.$$

[a] Yin Zhao and Daizhan Cheng. "On controllability and stabilizability of probabilistic Boolean control networks". In: *Science China Information Sciences* 57.1 (2014), pp. 1–14.

2.1 Definitions of Stability

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Definition 8 (Stability in Stochastic Sense (SSS))

A state $x_e \in \mathscr{D}_n$ is said to be stable in stochastic sense if[a]

$$\lim_{t\to\infty} \mathbb{E}[\vec{x}_t \mid x_0 = j] = \vec{x}_e \quad \forall j \in \mathscr{D}_n.$$

[a] Min Meng, Lu Liu, and Gang Feng. "Stability and l_1 gain analysis of Boolean networks with Markovian jump parameters". In: *IEEE Transactions on Automatic Control* 62.8 (2017), pp. 4222–4228.

Definition 9 (Stability in Distribution (SD))

A state $x_e \in \mathscr{D}_n$ is said to be stable in distribution if[a]

$$\lim_{t\to\infty} \mathbb{P}\left\{x_t = x_e \,\middle|\, x_0 = j\right\} = 1 \quad \forall j \in \mathscr{D}_n.$$

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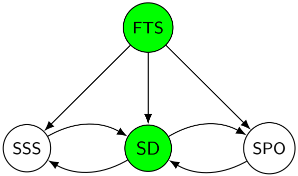

Relationship between different stabilities

- FTS and SD can be easily generalized to set stability.
- However, such generalizations of SPO and SSS are not convenient, because they require the existences of the limits $\lim_{t\to\infty} x_t$ and $\lim_{t\to\infty} \mathbb{E}x_t$, respectively.

Definition 10 (Finite-time Set Stability)

A subset $\mathcal{M} \subset \mathscr{D}_n$ is said to be finite-time stable if there is a positive integer T such that[a]

$$\mathbb{P}\{x_t \in \mathcal{M} \mid x_0 = j\} = 1 \quad \forall t \geq T, \forall j \in \mathscr{D}_n.$$

[a] Li Rui, Yang Meng, and Chu Tianguang. "概率布尔网络的集合镇定控制". In: 系统科学与数学 36.3 (2016), pp. 371–380.

Definition 11 (Set Stability in Distribution (SSD))

A subset $\mathcal{M} \subset \mathscr{D}_n$ is said to be stable in distribution if[a]

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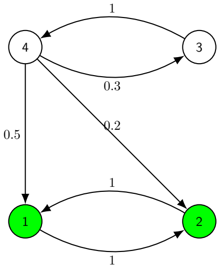
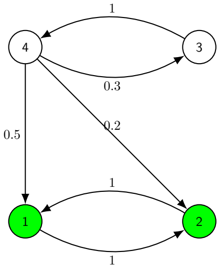


$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0.5 \\ 1 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathcal{M} = \{1, 2\}$$



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$$\mathcal{M} = \{1, 2\}$$

- The limitations

$$\lim_{t\to\infty} x(t), \quad \lim_{t\to\infty} \mathbb{E}\vec{x}(t)$$

  do not exist;

- However, for any x_0,

$$\lim_{t\to\infty} \mathbb{P}\left\{x(t) \in \mathcal{M} \mid x(0) = x_0\right\} = 1$$

- **Typical Set Stability Problem:** **Synchronization of networks**

  Consider two n-valued PLDSs

$$x_{t+1} = f(w_t, x_t), \quad z_{t+1} = g(v_t, z_t, x_t)$$

$$x_t, z_t \in \mathscr{D}_n$$

  - **Finite-time synchronization:** There exists a $T > 0$ such that

$$\mathbb{P}\{x_t = z_t \mid x_0 = j, z_0 = i\} = 1 \quad \forall t \geq T, \forall j, i \in \mathscr{D}_n.$$

  - **Asymptotical synchronization:**

$$\lim_{t \to \infty} \mathbb{P}\left\{x_t = z_t \,\middle|\, x_0 = j, z_0 = i\right\} = 1 \quad \forall j, i \in \mathscr{D}_n$$

.

The synchronization problem is equivalent to the stability of the combined system

$$\begin{cases} x_{t+1} = f(w_t, x_t) \\ z_{t+1} = g(v_t, z_t, x_t) \end{cases}$$

with respect to the synchronization set

$$\mathcal{M} := \{(j, j) \mid j \in \mathscr{D}_n\} \subset \mathscr{D}_n \times \mathscr{D}_n$$

Outline

2.2 Reachability-based Stability Analysis

Theorem 12

A PBN is finite-time stable with respect to x_e if and only if

$$\mathrm{Col}\left\{\mathbf{P}^{n-1}\right\} = \{\vec{x}_e\}, \quad (\textit{where } \mathbf{P} = L_f \ltimes \boldsymbol{p}^w) \tag{3}$$

Sketchy Proof: (Necessity) FT stability

2.2 Reachability-based Stability Analysis

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(Sufficiency) (3) $\Rightarrow$

$$\mathbf{P}\vec{x}_e = \mathbf{P}^n\vec{x}_0 = \mathbf{P}^{n-1}(\mathbf{P}\vec{x}_0) = [\vec{x}_e, \cdots, \vec{x}_e](\mathbf{P}\vec{x}_0) = \vec{x}_e$$

2.2 Reachability-based Stability Analysis

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$\Rightarrow$ x_e is a fixed point

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$\Rightarrow x_e$ is a fixed point$\Rightarrow$ For any $t \geq n$, any $j \in \mathscr{D}_n$,

$$\mathbb{P}\{x_t = x_e \mid x_0 = j\} \geq \mathbb{P}\{x(n-1) = x_e \mid x_0 = j\} = 1$$

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$\Rightarrow$ FT stability

- **Criterion of FT Stability in terms of STG**[4]

[4] Shiyong Zhu, Jianquan Lu, and Daniel W.C.Ho. "Finite-time Stability of Probabilistic Logical Networks: A Topological Sorting Approach". In: *IEEE Transactions on Circuits & Systems -II: Express Briefs* 67.4 (2020), pp. 695–699.

2.2 Reachability-based Stability Analysis

- **Criterion of FT Stability in terms of STG**[4]

$$\mathbb{P}\{x_t = x_e \mid x_0 = j\} = 1 \quad \forall t \geq T, \forall j \in \mathscr{D}_n.$$

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$$\Updownarrow$$

$$\begin{cases} \text{(i) } x_e \text{ is a fixed point} \\ \text{(ii) } x_0 \to x_e \quad \forall x_0 \\ \text{(iii) Any path from any } x_0 \text{ to } x_e \text{ in } \mathcal{G} \setminus (x_e, x_e) \text{ is with finite length} \end{cases}$$

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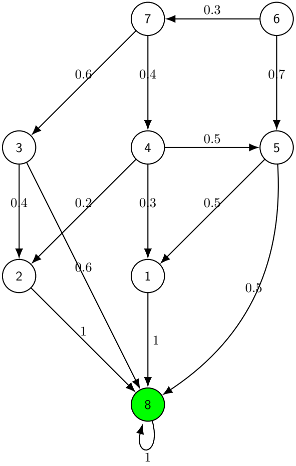
$$\Updownarrow$$

$$\mathcal{G} \setminus (x_e, x_e) \text{ is acyclic}$$

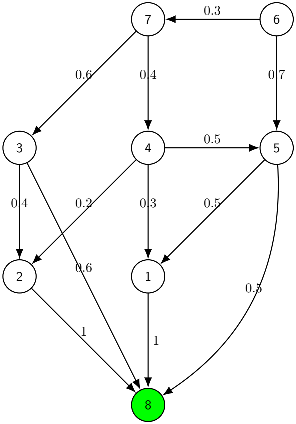
- **Note:** $\mathcal{G} \setminus (x_e, x_e)$ is the graph obtained from the STG $\mathcal{G}$ of the PLDS by removing the self-loop of x_e

[4] Shiyong Zhu, Jianquan Lu, and Daniel W.C.Ho. "Finite-time Stability of Probabilistic Logical Networks: A Topological Sorting Approach". In: *IEEE Transactions on Circuits & Systems -II: Express Briefs* 67.4 (2020), pp. 695–699.

2.2 Reachability-based Stability Analysis



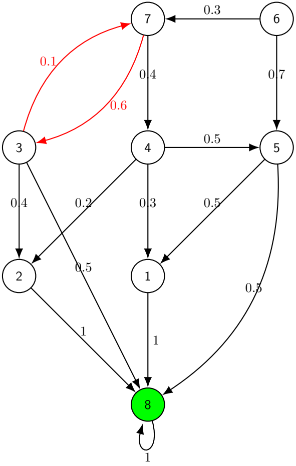
STG $\mathcal{G}$



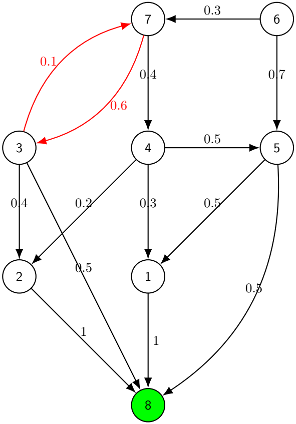
$\mathcal{G} \setminus (x_e, x_e)$

STG of a PLDS that is finite-time stable w.r.t. $x_e = 8$

2.2 Reachability-based Stability Analysis



STG $\mathcal{G}$



$\mathcal{G} \setminus (x_e, x_e)$

STG of a PLDS that is not finite-time stable w.r.t. $x_e = 8$

Theorem 13

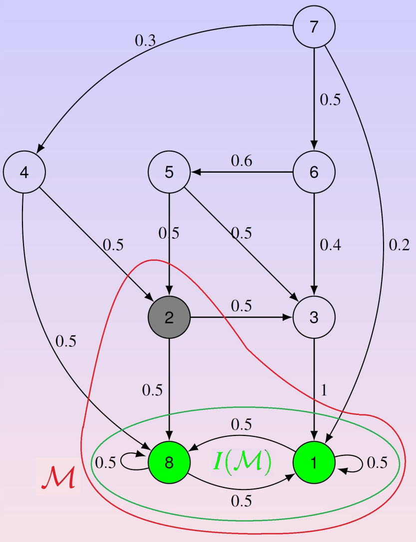
A PBN is finite-time stable with respect to x_e if and only if $\mathcal{G} \setminus (x_e, x_e)$ is acyclic[a].

[a] Shiyong Zhu, Jianquan Lu, and Daniel W.C.Ho. "Finite-time Stability of Probabilistic Logical Networks: A Topological Sorting Approach". In: *IEEE Transactions on Circuits & Systems -II: Express Briefs* 67.4 (2020), pp. 695–699.

- **Finite-time Set Stability**
  - Finite-time stability w.r.t. $\mathcal{M}$

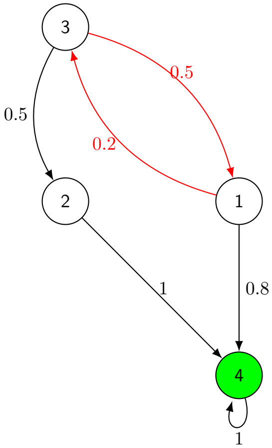
    $\Leftrightarrow$ Finite-time stability w.r.t. the largest invariant subset $I(\mathcal{M})$ in $\mathcal{M}$

    $\Leftrightarrow$ $I(\mathcal{M}) \neq \emptyset$ and the STG has no cycles outside $I(\mathcal{M})$.

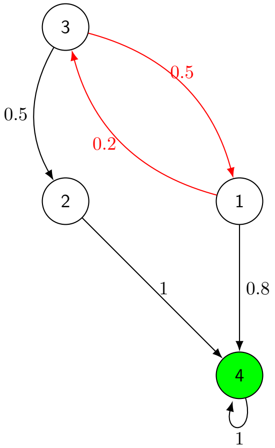


2.2 Reachability-based Stability Analysis

- **An asymptotically stable PLDS that is not FT stable**



- **An asymptotically stable PLDS that is not FT stable**



$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0.8 & 1 & 0 & 1 \end{bmatrix}$$

$$\lim_{t\to\infty} \mathbb{P}\{x_t = 4 \mid x_0 = j\}$$

$$= \lim_{t\to\infty} [\mathbf{P}^t]_{4,j} = 1 \quad \forall j$$

- **Criterion of Stability with Probability One**

$$\mathbb{P}\left\{\lim_{t\to\infty} x_t = x_e \mid x_0 = j\right\} = 1 \quad \forall j \in \mathscr{D}_n.$$

$$\Updownarrow$$

$$\begin{cases} x_e \text{ is a fixed point. (Thus, it is recurrent)} \\ x_0 \to x_e \quad \forall x_0. \end{cases}$$

Theorem 14

A PLDS is asymptotically stable w.r.t. $x_e = i$ with probability one if and only if x_e is a fixed point and[a]

$$\mathrm{Row}_i\left(\mathbf{P}^{n-1}\right) \succ 0 \tag{4}$$

[a] Yin Zhao and Daizhan Cheng. "On controllability and stabilizability of probabilistic Boolean control networks". In: *Science China Information Sciences* 57.1 (2014), pp. 1–14.

2.2 Reachability-based Stability Analysis

- **Criterion of asymptotical stability in distribution**

Theorem 15

A PLDS is asymptotically stable w.r.t. x_e in distribution if and only if[a]

$$\begin{cases} x_e \text{ is a fixed point.} \\ x_0 \to x_e \quad \forall x_0. \end{cases}$$

Or, equivalently, x_e is a fixed point and $\mathrm{Row}_i\left(\mathbf{P}^{n-1}\right) \succ 0$.

[a] Yuqian Guo et al. "Stability and Set Stability in Distribution of Probabilistic Boolean Networks". In: *IEEE Transactions on Automatic Control* 64 (2 2019), pp. 736–742.

- The necessity is obvious. A sketchy proof for sufficiency is provided in the next page.

Sketchy Proof of Sufficiency.

$$\lim_{t\to\infty} \mathbb{P}\left\{x_t = x_e \,\middle|\, x_0 = j\right\} = 1 \quad \forall j \in \mathscr{D}_n.$$

$$\Updownarrow$$

$$\lim_{t\to\infty} \mathbf{P}^t = \begin{bmatrix} 0_{(n-1)\times n} \\ \mathbf{1}_n^\top \end{bmatrix} \quad \text{(Assume } x_e = n\text{)}$$

$$\Updownarrow$$

$$\lim_{t\to\infty} \boldsymbol{\alpha}_t = \mathbf{1}_{n-1}, \quad \text{where} \quad \mathbf{P}^t := \begin{bmatrix} \boldsymbol{\Gamma}_t^\top & 0_{(n-1)\times 1} \\ \boldsymbol{\alpha}_t^\top & 1 \end{bmatrix}.$$

$$\Updownarrow$$

$$\lim_{t\to\infty} (\underbrace{\boldsymbol{\alpha}_{nt} - \mathbf{1}_{n-1}}_{\boldsymbol{\eta}_t}) = 0 \quad \text{(By Monotonicity)}$$

2.2 Reachability-based Stability Analysis

$$\mathbf{P}(n(t+1)) = \mathbf{P}(nt)\mathbf{P}(n)$$

$$\Downarrow$$

$$\boldsymbol{\alpha}_{n(t+1)} = \boldsymbol{\Gamma}_n \boldsymbol{\alpha}_{nt} + \boldsymbol{\alpha}_n$$

$$\Downarrow$$

$$\boldsymbol{\alpha}_{n(t+1)} - \mathbf{1}_{n-1} = \boldsymbol{\Gamma}_n(\boldsymbol{\alpha}_{nt} - \mathbf{1}_{n-1}) + \underbrace{\boldsymbol{\Gamma}_n \mathbf{1}_{n-1} + \boldsymbol{\alpha}_n - \mathbf{1}_{n-1}}_{=0}$$

$$\Downarrow$$

$$\boldsymbol{\eta}_{t+1} = \boldsymbol{\Gamma}_n \boldsymbol{\eta}_t$$

2.2 Reachability-based Stability Analysis

$$\mathbf{P}(n(t+1)) = \mathbf{P}(nt)\mathbf{P}(n)$$
$$\Downarrow$$
$$\boldsymbol{\alpha}_{n(t+1)} = \boldsymbol{\Gamma}_n\boldsymbol{\alpha}_{nt} + \boldsymbol{\alpha}_n$$
$$\Downarrow$$
$$\boldsymbol{\alpha}_{n(t+1)} - \mathbf{1}_{n-1} = \boldsymbol{\Gamma}_n(\boldsymbol{\alpha}_{nt} - \mathbf{1}_{n-1}) + \underbrace{\boldsymbol{\Gamma}_n\mathbf{1}_{n-1} + \boldsymbol{\alpha}_n - \mathbf{1}_{n-1}}_{=0}$$
$$\Downarrow$$
$$\boldsymbol{\eta}_{t+1} = \boldsymbol{\Gamma}_n\boldsymbol{\eta}_t$$

$$\begin{cases} x_e \text{ is a fixed point.} \\ x_0 \to x_e \quad \forall x_0 \end{cases}$$
$$\Downarrow$$
$$\boldsymbol{\alpha}_n \succ 0$$
$$\Downarrow$$
$\boldsymbol{\Gamma}_n$ is strictly Schur stable

2.2 Reachability-based Stability Analysis

$$\mathbf{P}(n(t+1)) = \mathbf{P}(nt)\mathbf{P}(n)$$

$$\Downarrow$$

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$$\Downarrow$$

$$\boldsymbol{\alpha}_{n(t+1)} - \mathbf{1}_{n-1} = \boldsymbol{\Gamma}_n(\boldsymbol{\alpha}_{nt} - \mathbf{1}_{n-1}) + \underbrace{\boldsymbol{\Gamma}_n \mathbf{1}_{n-1} + \boldsymbol{\alpha}_n - \mathbf{1}_{n-1}}_{=0}$$

$$\Downarrow$$

$$\boldsymbol{\eta}_{t+1} = \boldsymbol{\Gamma}_n \boldsymbol{\eta}_t$$

$$\begin{cases} x_e \text{ is a fixed point.} \\ x_0 \to x_e \quad \forall x_0 \end{cases}$$

$$\Downarrow$$

$$\boldsymbol{\alpha}_n \succ 0$$

$$\Downarrow$$

$\boldsymbol{\Gamma}_n$ is strictly Schur stable

$$\Downarrow$$

$$\lim_{t\to\infty} \boldsymbol{\eta}_t = 0 \quad \Rightarrow \quad \lim_{t\to\infty} \boldsymbol{\alpha}_t = \mathbf{1}_{n-1}$$

- **Criterion of asymptotical stability in stochastic sense**

$$\lim_{t\to\infty} \mathbb{E}[\vec{x}_t \mid x_0 = j] = \vec{x}_e \quad \forall j \in \mathscr{D}_n.$$

$$\Updownarrow \quad \mathbb{E}[\vec{x}_t \mid x_0 = j] = \mathrm{Col}_j[\mathbf{P}^t]$$

$$\lim_{t\to\infty} \mathrm{Col}_j[\mathbf{P}^t] = \vec{x}_e \quad \forall j \in \mathscr{D}_n$$

$$\Updownarrow$$

Asymptotically stable in distribution

Note: The above results confirm that SSO, SSS, and SD are equivalent indeed.

2.2 Reachability-based Stability Analysis

Corollary 16

Consider two PLDSs of the same size with TPMs $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively. Suppose that x_e is the fixed point of both PLDSs, that is,

$$\mathbf{P}_1\vec{x}_e = \mathbf{P}_2\vec{x}_e = \vec{x}_e.$$

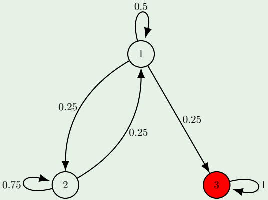
- *Suppose that $\mathbf{P}_1 \sqsubseteq \mathbf{P}_2$. If PLDS $\mathbf{P}_1$ is asymptotical x_e-stable, then, so is PLDS $\mathbf{P}_2$.*
- *Suppose that $\mathbf{P}_1 \sim_h \mathbf{P}_2$. Then, PLDS $\mathbf{P}_1$ is asymptotical x_e-stable iff PLDS $\mathbf{P}_2$ is asymptotical x_e-stable.*

Example 17

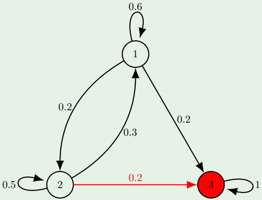
The STGs corresponding the three TPMs satisfying ($x_e = 3$)

$$\mathbf{P}_1\vec{x}_e = \mathbf{P}_2\vec{x}_e = \mathbf{P}_3\vec{x}_e = \vec{x}_e,$$

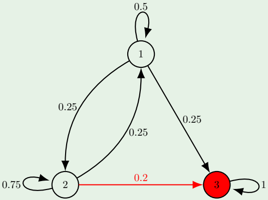
$$\mathbf{P}_1 \sqsubseteq \mathbf{P}_2 \sim_h \mathbf{P}_3.$$



$\mathbf{P}_1$



$\mathbf{P}_2$



$\mathbf{P}_3$

- **Asymptotical Set Stability**

$$\lim_{t\to\infty} \mathbb{P}\left\{x_t \in \mathcal{M} \;\middle|\; x_0 = j\right\} = 1 \quad \forall j \in \mathscr{D}_n.$$

$$\Updownarrow$$

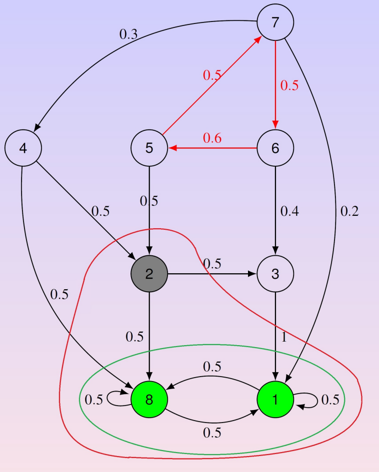
$$\lim_{t\to\infty} \mathbb{P}\left\{x_t \in I(\mathcal{M}) \;\middle|\; x_0 = j\right\} = 1 \quad \forall j \in \mathscr{D}_n.$$

$$\Updownarrow$$

$$\begin{cases} I(\mathcal{M}) \neq \emptyset \\ x_0 \to I(\mathcal{M}) \quad \forall x_0 \end{cases}$$

Note: $x_0 \to I(\mathcal{M})$ means $x_0 \to x$ for some $x \in I(\mathcal{M})$.

2.2 Reachability-based Stability Analysis



STG of a asymptotically $\mathcal{M}$-stable PLDS that is not finite-time stable.

Outline

- **Dynamics of State PDV $\boldsymbol{\pi}_t := \mathbb{E}\vec{x}_t$**

$$\boldsymbol{\pi}_{t+1} = \mathbf{P}\boldsymbol{\pi}_t, \quad \boldsymbol{\pi}_0 = \vec{x}_0 \in \Delta_n. \tag{5}$$

  - **Note:** The PLDS is asymptotically x_e-stable iff

$$\lim_{t\to\infty} \boldsymbol{\pi}_t = \vec{x}_e, \quad \forall \boldsymbol{\pi}_0 \in \Delta_n.$$

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$$\boldsymbol{\pi}_{t+1} = \mathbf{P}\boldsymbol{\pi}_t, \quad \boldsymbol{\pi}_0 = \vec{x}_0 \in \Delta_n. \tag{5}$$

  - **Note:** The PLDS is asymptotically x_e-stable iff

$$\lim_{t\to\infty} \boldsymbol{\pi}_t = \vec{x}_e, \quad \forall \boldsymbol{\pi}_0 \in \Delta_n.$$

- **Error System:** We define the state distribution error as

$$\boldsymbol{e}_t := \boldsymbol{\pi}_t - \vec{x}_e$$

If x_e is a fixed point, then,

$$\boldsymbol{e}_{t+1} = \mathbf{P}\boldsymbol{e}_t, \quad \boldsymbol{e}_0 \in \Delta_n - \vec{x}_e,$$

where $\Delta_n - \vec{x}_e := \{\delta_n^j - \vec{x}_e \mid j \in \mathscr{D}_n\}$.

- $n-1$-**dimensional invariant subspace of error system:** We define

$$\boldsymbol{\alpha}_i := \delta_n^i - \delta_n^{x_e}, \quad i \in [1:n].$$

We construct an $n \times (n-1)$ matrix as

$$\mathbf{M}_{x_e} := [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_{x_e-1}, \boldsymbol{\alpha}_{x_e+1}, \cdots, \boldsymbol{\alpha}_n].$$

Then $\mathbf{M}_{x_e}$ is of full column rank. We define

$$\mathcal{M}_{x_e} := \mathrm{Span}\{\Delta_n - \vec{x}_e\} = \mathrm{Span}\{\mathbf{M}_{x_e}\}.$$

  - By the linearity, the error system

$$\boldsymbol{e}_{t+1} = \mathbf{P}\boldsymbol{e}_t, \quad \boldsymbol{e}_0 \in \Delta_n - \vec{x}_e$$

is finite-time/asymptotically stable iff the following system is finite-time/asymptotically stable:

$$\boldsymbol{e}_{t+1} = \mathbf{P}\boldsymbol{e}_t, \quad \boldsymbol{e}_0 \in \mathcal{M}_{x_e}$$

Lemma 18

If x_e is a fixed point, then, $\mathcal{M}_{x_e}$ is an $(n-1)$-dimensional invariant subspace of

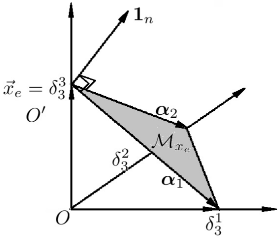
$$\boldsymbol{e}_{t+1} = \mathbf{P}\boldsymbol{e}_t$$

Proof:

- $\mathbf{1}_n$ is orthogonal to each $\boldsymbol{\alpha}_i$, $i \in [1:n] \setminus \{x_e\}$. Thus, it is orthogonal to $\mathcal{M}_{x_e}$.
- For any $\boldsymbol{e}_0 \in \mathcal{M}_{x_e}$ and any t, $\boldsymbol{e}_t = \mathbf{P}^t \boldsymbol{e}_0$ and

$$\mathbf{1}_n^\top \boldsymbol{e}_t = \underbrace{\mathbf{1}_n^\top \mathbf{P}^t}_{=\mathbf{1}_n^\top} \boldsymbol{e}_0 = \mathbf{1}_n^\top \boldsymbol{e}_0 = 0.$$

Thus, $\boldsymbol{e}_t$ is orthogonal to $\mathbf{1}_n$ and $\boldsymbol{e}_t \in \mathcal{M}_{x_e}$.



$n = 3$, $x_e = 3$

$$\mathcal{M}_{x_e} = \mathrm{Span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}\right\}$$

- **Restriction of error system on $\mathcal{M}_{x_e}$**
  - We define the coordinate transformation as

$$\boldsymbol{e}_t = [\mathbf{M}_{x_e}, \mathbf{1}_n] \begin{bmatrix} \boldsymbol{z}_1(t) \\ z_2(t) \end{bmatrix} = \mathbf{M}_{x_e} \boldsymbol{z}_1(t) + \mathbf{1}_n z_2(t)$$

where $\boldsymbol{z}_1(t) \in \mathbb{R}^{n-1}$, $z_2(t) \in \mathbb{R}$. Then,

$$\begin{bmatrix} \boldsymbol{z}_1(t+1) \\ z_2(t+1) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{x_e}^{+} \mathbf{P} \mathbf{M}_{x_e} & \mathbf{M}_{x_e}^{+} \mathbf{P} \mathbf{1}_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_1(t) \\ z_2(t) \end{bmatrix}$$

where $\mathbf{M}_{x_e}^{+} := (\mathbf{M}_{x_e}^{\top} \mathbf{M}_{x_e})^{-1} \mathbf{M}_{x_e}^{\top}$ is the pseudo-inverse of $\mathbf{M}_{x_e}$.
  - In the z-coordinate system, $\mathcal{M}_{x_e} = \{(\boldsymbol{z}_1^{\top}, z_2)^{\top} \in \mathbb{R}^n \mid z_2 = 0\}$. By letting $z_2(t) = 0$,

$$\boldsymbol{z}_1(t+1) = \mathbf{M}_{x_e}^{+} \mathbf{P} \mathbf{M}_{x_e} \boldsymbol{z}_1(t)$$

2.3 Error-based Stability Analysis

Theorem 19

The PLDS is finite-time x_e-stable iff

- x_e *is a fixed point.*
- *The* $(n-1) \times (n-1)$ *matrix* $\mathbf{D} := \mathbf{M}_{x_e}^{+}\mathbf{P}\mathbf{M}_{x_e}$ *is nipolent.*

Theorem 20

The PLDS is asymptotically x_e-stable iff[a]

- x_e *is a fixed point.*
- *The* $(n-1) \times (n-1)$ *matrix* $\mathbf{D} := \mathbf{M}_{x_e}^{+} \mathbf{P} \mathbf{M}_{x_e}$ *is Schur stable.*

[a] Guo Yuqian et al. "Asymptotical Stabilization of Logic Dynamical Systems via Output-Based Random Control". In: *IEEE transactions on Automatic Control* 69.5 (2024), pp. 3286 –3293.

Theorem 20

The PLDS is asymptotically x_e-stable iff[a]

- x_e *is a fixed point.*
- *The* $(n-1)\times(n-1)$ *matrix* $\mathbf{D} := \mathbf{M}_{x_e}^{+}\mathbf{PM}_{x_e}$ *is Schur stable.*

[a] Guo Yuqian et al. "Asymptotical Stabilization of Logic Dynamical Systems via Output-Based Random Control". In: *IEEE transactions on Automatic Control* 69.5 (2024), pp. 3286 –3293.

Remark 1

Suppose Q *is an* $(n-1)\times(n-1)$ *positive-definite matrix. Then, by according to Theorem 20, the PLDS is asymptotically* x_e*-stable iff there exists an* $(n-1)\times(n-1)$ *positive-definite matrix* Ω *such that*

$$\mathbf{D}^\top \Omega \mathbf{D} - \Omega = -Q.$$

Outline

Outline

3. State Feedback Stabilization

- Consider a PLDS

$$x_{t+1} = f(w_t, u_t, x_t)$$

and its algebraic form

$$\vec{x}_{t+1} = L_f \ltimes \vec{w}_t \ltimes \vec{u}_t \ltimes \vec{x}_t$$

  - $x_t \in \mathscr{D}_n$, $u_t \in \mathscr{D}_m$, $y_t \in \mathscr{D}_q$
  - $f : \mathscr{D}_{n_w} \times \mathscr{D}_m \times \mathscr{D}_n \to \mathscr{D}_n$;
  - $w_t \sim \boldsymbol{p}^w$;
  - $L_f \in \mathscr{L}_{n \times n_w mn}$
  - TPMs $\mathbf{P} = L_f \ltimes \boldsymbol{p}^w$, $\mathbf{P}_j = L_f \ltimes \boldsymbol{p}^w \ltimes \delta_m^j$.

- **Closed-loop TPM under State Feedback**

$$\vec{u}_t = K\vec{x}_t, \quad K \in \mathscr{L}_{m\times n}$$

$$\Downarrow$$

$$\begin{aligned} \vec{x}_{t+1} &= L_f \ltimes \vec{w}_t \ltimes \vec{u}_t \ltimes \vec{x}_t \\ &= L_f \ltimes \vec{w}_t \ltimes K \ltimes \vec{x}_t \ltimes \vec{x}_t \\ &= L_f \ltimes \vec{w}_t \ltimes K\mathbf{R}_{[n]}\vec{x}_t \end{aligned}$$

$\mathbf{R}_{[n]}$: Power-reducing Matrix

$$\Downarrow$$

$$\mathbf{P}_K = (L_f \ltimes \boldsymbol{p}^w)K\mathbf{R}_{[n]}.$$

3. State Feedback Stabilization

- Problem: Find a state-feedback

$$u(t) = Kx(k)$$

to stabilize a PBN to a point or a subset in finite-time or asymptotically.

- If

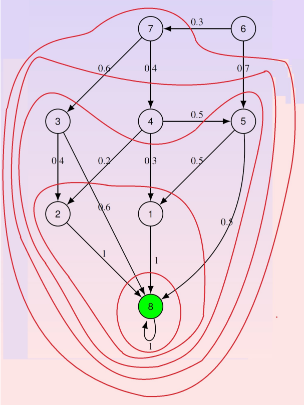
$$K = \delta_m[k_1, k_2, \cdots, k_{2^n}]$$

Then, the TPM of the closed loop, denoted by $\mathbf{P}_K$, is

$$\mathrm{Col}_j(\mathbf{P}_K) = \mathrm{Col}_j(\mathbf{P}_{k_j})$$

Outline

- **Hierarchical structure of the STG of a Finite-time stable PLDS**



$$\Omega_0 = \{x_e\}$$

$$\Omega_1 = \{x \mid \mathbb{P}\{x_{t+1} \in \Omega_0 \mid x_t = x\} = 1\}$$

$$\Omega_k = \{x \mid \mathbb{P}\{x_{t+1} \in \Omega_{k-1} \mid x_t = x\} = 1\}$$

- We can always rearrange the STG into the hierarchical structure for a finite-time stable PLDS.

- **Finite-time Stablizability by State Feedback**
  - Define a sequence of subsets as

$$\begin{cases} \Omega_0 = \{x_e\} \\ \Omega_k = \left\{x \,\middle|\, \exists u \text{ s.t. } \mathbb{P}\{x_{t+1} \in \Omega_{k-1} \,\middle|\, x_t = x, u_t = u\} = 1\right\} \\ \quad k = 1, 2, 3, \cdots \end{cases}$$

  - If x_e is control invariant, then $\Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \cdots$

Theorem 21

A PLDS is finite-time stabilizable w.r.t. x_e by a state feedback iff[a]

- *x_e is control invariant;*
- *There is a positive integer $K \leq n - 1$ such that $\Omega_K = \mathscr{D}_n$.*

[a] Rui Li, Meng Yang, and Tianguang Chu. "State feedback stabilization for probabilistic Boolean networks". In: *Automatica* 50.4 (2014), pp. 1272–1278.

- **Design of Finite-time Stabilizing feedback gain**[5]
  - Assigning a control $u(x_e)$ for x_e such that

$$\mathbb{P}\{x_{t+1} = x_e \mid x_t = x_e\} = 1;$$

  - Assigning a control $u(x)$ for every $x \in \Omega_k \setminus \Omega_{k-1}$ such that

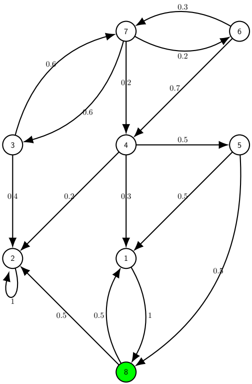
$$\mathbb{P}\{x_{t+1} \in \Omega_{k-1} \mid x_t = x\} = 1.$$

  - Then,

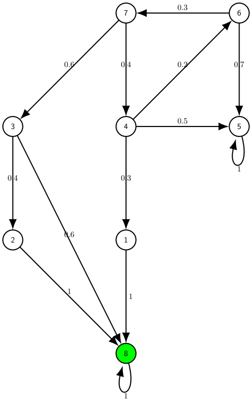
$$K = \delta_m[u(1), u(2), \cdots, u(n)]$$

[5] Rui Li, Meng Yang, and Tianguang Chu. "State feedback stabilization for probabilistic Boolean networks". In: *Automatica* 50.4 (2014), pp. 1272–1278.

3.1 Finite-time Stabilization by State Feedback



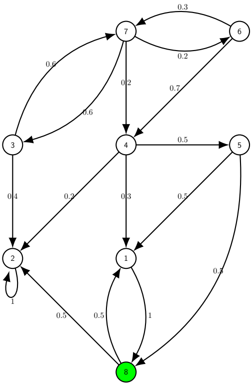
$u \equiv 1$



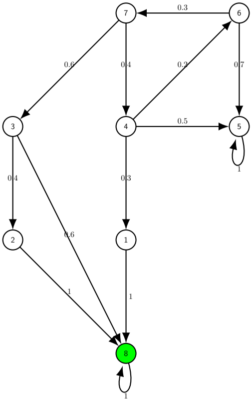
$u \equiv 2$

$$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$$

3.1 Finite-time Stabilization by State Feedback

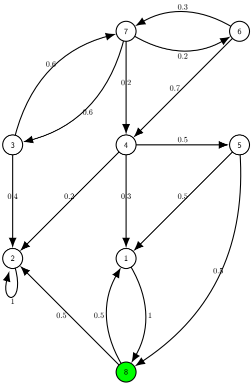


$u \equiv 1$

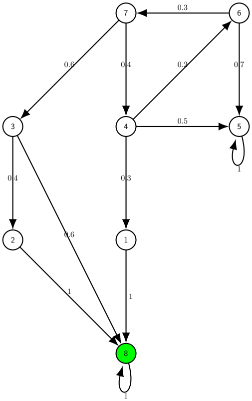


$u \equiv 2$

$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$



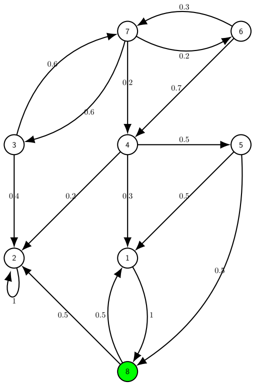
$u \equiv 1$



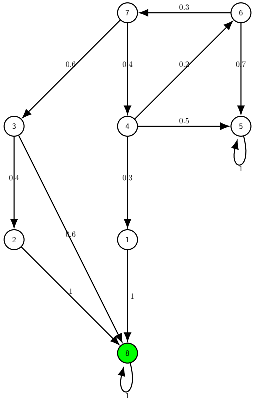
$u \equiv 2$

$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$

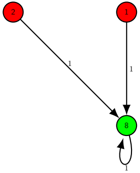
3.1 Finite-time Stabilization by State Feedback



$u \equiv 1$

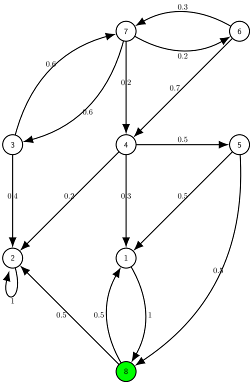


$u \equiv 2$

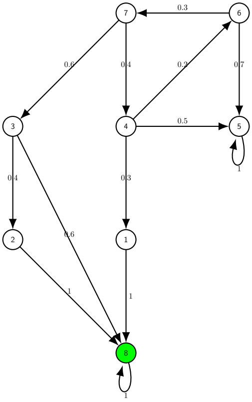


$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$

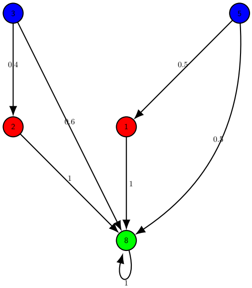
3.1 Finite-time Stabilization by State Feedback



$u \equiv 1$

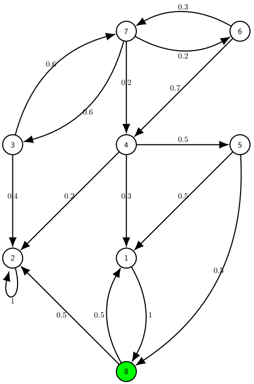


$u \equiv 2$

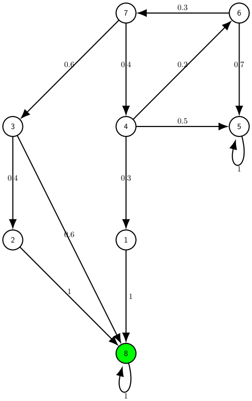


$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$

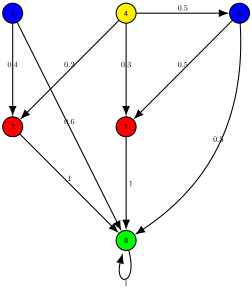
3.1 Finite-time Stabilization by State Feedback



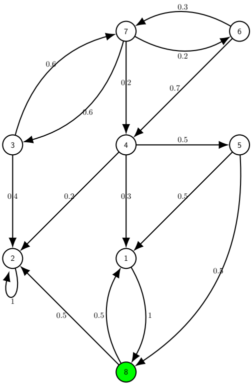
$u \equiv 1$



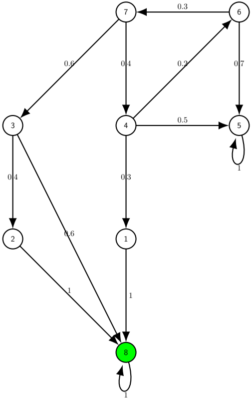
$u \equiv 2$



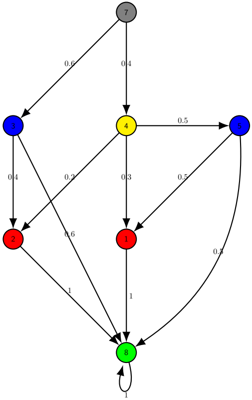
$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$



$u \equiv 1$

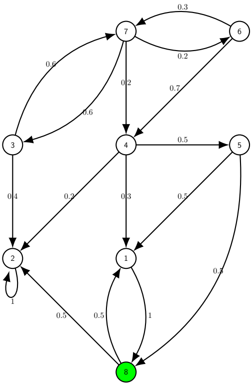


$u \equiv 2$

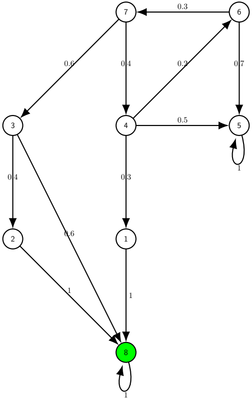


$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$

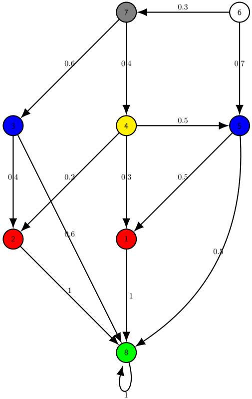
3.1 Finite-time Stabilization by State Feedback



$u \equiv 1$



$u \equiv 2$



$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$

3.1 Finite-time Stabilization by State Feedback

- Finite-time Feedback Set Stabilization

Finite-time Feedback $\mathcal{M}$-Stabilizable

$$\Updownarrow$$

Finite-time Feedback $I_c(\mathcal{M})$-Stabilizable

Outline

- **Asymptotical Feedback Stabilizability**

Theorem 22

A state x_e is asymptotically feedback stabilizable iff[a b]

1. x_e *is a control-fixed point, and*
2. $x_0 \xrightarrow{u} x_e \ \forall x_0$, *that is,*

$$\vec{x}_e^\top \left(\mathbf{P} \ltimes \mathbf{1}_m\right)^{n-1} \succ 0.$$

[a] Rongpei Zhou et al. "Asymptotical Feedback Set Stabilization of Probabilistic Boolean Control Networks". In: *IEEE Transactions on Neural Network & Learning Systems* 31.11 (2020), pp. 4524–4537.

[b] Wang Liqing et al. "Stabilization and Finite-Time Stabilization of Probabilistic Boolean Control Networks". In: *IEEE Transactions on Systems, Man, and Cybernetics: Systems* 51.3 (2021), pp. 1559–1566.

- **Asymptotical Feedback Set Stabilizability**

Theorem 23

A subset $\mathcal{M}$ is asymptotically feedback stabilizable iff[a]

1. $I_c(\mathcal{M}) \neq \emptyset$, *and*
2. $x_0 \xrightarrow{u} I_c(\mathcal{M})\ \forall x_0$, *that is,*

$$\sum_{j \in I_c(\mathcal{M})} \mathrm{Row}_j \left[(\mathbf{P} \ltimes \mathbf{1}_m)^{n-1} \right] \succ 0.$$

[a] Rongpei Zhou et al. "Asymptotical Feedback Set Stabilization of Probabilistic Boolean Control Networks". In: *IEEE Transactions on Neural Network & Learning Systems* 31.11 (2020), pp. 4524–4537.

- **Design of Asymptotically Stabilizing Feedback**

  - Decomposition of State Space:

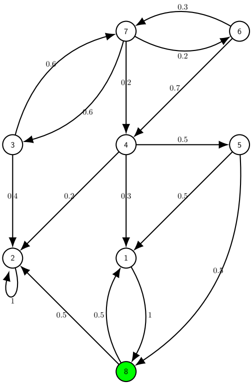
$$\begin{cases} \Theta_0 = I_c(\mathcal{M}), \\ \Theta_k = \left\{ j \in \left(\bigcup_{s=0}^{k-1} \Theta_s \right)^c \,\middle|\, \sum_{i \in \Theta_{k-1}} [\mathbf{P} \ltimes \mathbf{1}_m]_{i,j} > 0 \right\}, \\ \quad k = 1, 2, \cdots, \lambda. \end{cases}$$

  - For any $j \in \mathscr{D}_n$, there is a unique k_j such that $j \in \Theta_{k_j}$. Then, we assign state j a control u_j as

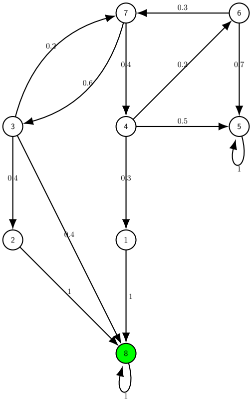
$$\sum_{i \in \Theta_{k_j - 1}} [\mathbf{P} \ltimes \delta_m^{u_j}]_{i,j} > 0 \quad \text{where } \Theta_{-1} := \Theta_0$$

  - Stabilizing state feedback gain: $K = \delta_m[u_1, u_2, \cdots, u_n]$.

3.2 Asymptotical Stabilization by State Feedback



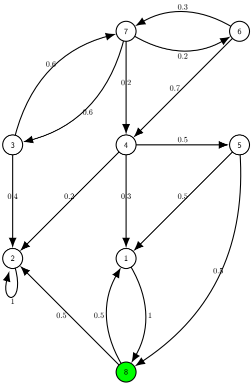
$u \equiv 1$



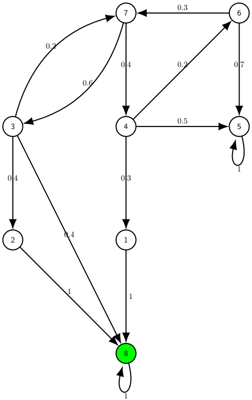
$u \equiv 2$

$$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$$

3.2 Asymptotical Stabilization by State Feedback

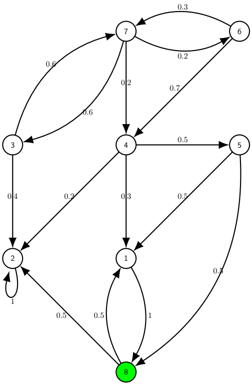


$u \equiv 1$

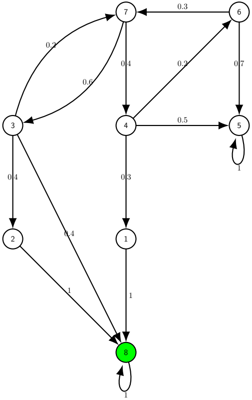


$u \equiv 2$

$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$



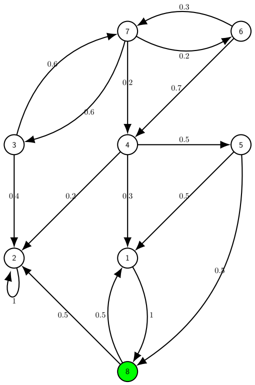
$u \equiv 1$



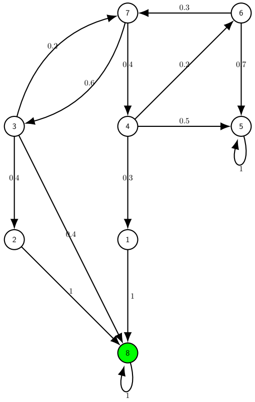
$u \equiv 2$

$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$

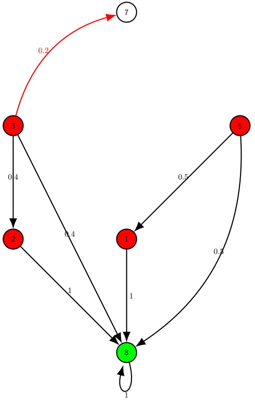
3.2 Asymptotical Stabilization by State Feedback



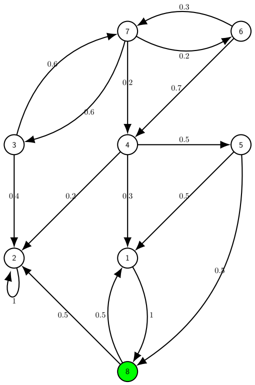
$u \equiv 1$



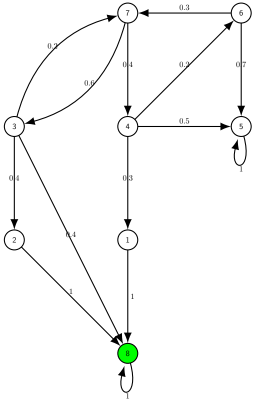
$u \equiv 2$



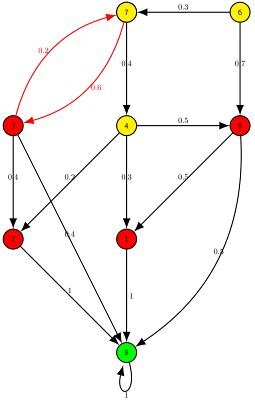
$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$



$u \equiv 1$



$u \equiv 2$



$\vec{u}_t = \delta_2[1, 2, 2, 1, 1, 2, 2, 2]\vec{x}_t$

Outline

Outline

Outline

- **Logic Dynamical System in Algebraic Form**

$$\begin{cases} \vec{x}_{t+1} = L \ltimes \vec{w}_t \ltimes \vec{u}_t \ltimes \vec{x}_t \\ \vec{y}_t = H\vec{x}_t \end{cases} \tag{6}$$

  - $x_t \in \mathscr{D}_n$, $u_t \in \mathscr{D}_m$, and $y_t \in \mathscr{D}_q$
  - $\omega_t \sim \boldsymbol{p}^\omega \in \Upsilon_N$

- **Deterministic output feedback**[6][7][8]

$$\vec{u}_t = F\vec{y}_t, \quad F \in \mathscr{L}_{m\times q}$$

  - The deterministic output feedback has a limitation (See the next page)

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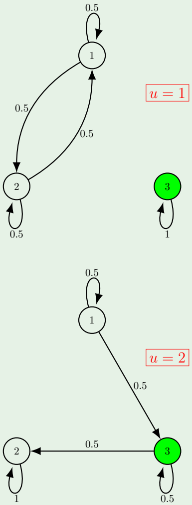
[6] Nicoletta Bof, Ettore Fornasini, and Maria Elena Valcher. "Output feedback stabilization of Boolean control networks". In: *Automatica* 57 (2015), pp. 21–28.

[7] Haitao Li and Yuzhen Wang. "Output feedback stabilization control design for Boolean control networks". In: *Automatica* 49.12 (2013), pp. 3641–3645.

[8] Rongjian Liu et al. "Output feedback control for set stabilization of Boolean control networks". In: *IEEE transactions on neural networks and learning systems* 31.6 (2019), pp. 2129 –2139.

4.1 Deterministic and Random Output Feedback

Example 24 (A Motivating Example)



- Consider a PLDS

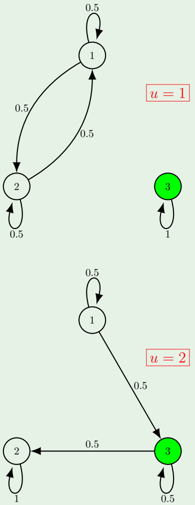
$$\begin{cases} \vec{x}_{t+1} = L \ltimes \vec{\omega}_t \ltimes \vec{u}_t \ltimes \vec{x}_t \\ \vec{y}_t = H\vec{x}_t \end{cases}$$

$$L = \delta_3[1,2,3,3,2,2,2,1,3,1,2,3]$$

$$\omega_t \sim \boldsymbol{p}^\omega = [0.5,\ 0,5]^\top, \quad H = \delta_2[1,1,2]$$

  - $x_e =$ is unstabilizable by any time-invariant deterministic output feedback.

Example 24 (A Motivating Example)



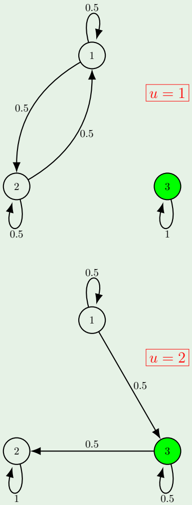
- Consider a PLDS

$$\begin{cases} \vec{x}_{t+1} = L \ltimes \vec{\omega}_t \ltimes \vec{u}_t \ltimes \vec{x}_t \\ \vec{y}_t = H \vec{x}_t \end{cases}$$

$$L = \delta_3[1,2,3,3,2,2,2,1,3,1,2,3]$$
$$\omega_t \sim \boldsymbol{p}^{\omega} = [0.5,\ 0,5]^{\top}, \quad H = \delta_2[1,1,2]$$

  - $x_e =$ is unstabilizable by any time-invariant deterministic output feedback.
  - **Is it a stabilizing time-invariant output feedback?**

Example 24 (A Motivating Example)



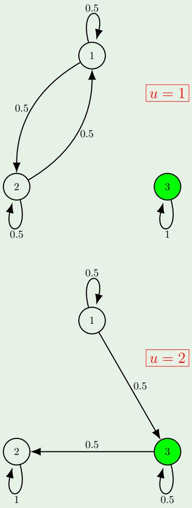
- Consider a PLDS

$$\begin{cases} \vec{x}_{t+1} = L \ltimes \vec{\omega}_t \ltimes \vec{u}_t \ltimes \vec{x}_t \\ \vec{y}_t = H\vec{x}_t \end{cases}$$

$$L = \delta_3[1, 2, 3, 3, 2, 2, 2, 1, 3, 1, 2, 3]$$
$$\omega_t \sim \boldsymbol{p}^{\omega} = [0.5,\ 0,5]^{\top}, \quad H = \delta_2[1, 1, 2]$$

  - $x_e =$ is unstabilizable by any time-invariant deterministic output feedback.
  - **Is it a stabilizing time-invariant output feedback? Yes!**

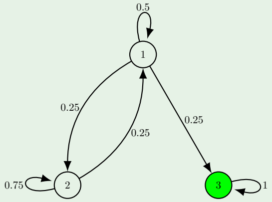
Example 25 (Example 24 Revisited)



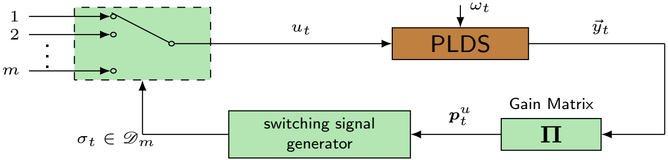
- We apply the following control strategy:

$$u_t \sim \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix} \vec{y}_t.$$

  - At each t, u_t is randomly selected from $\mathscr{D}_2$ according to the above distribution.
  - The closed-loop is a homogeneous Markovian chain and is asymptotically stable w.r.t. 3.



$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0.25 & 0 & 1 \end{bmatrix}$$



- **Random Output Feedback**

$$u_t \sim \mathbf{\Pi}\vec{y}_t$$

  - Each column of $\mathbf{\Pi} \in \mathbb{R}^{m\times q}$ is a PDV satisfying $\mathbf{\Pi} \succeq 0$, $\mathbf{1}_m^\top \mathbf{\Pi} = \mathbf{1}_q^\top$.
  - Deterministic output feedback $\vec{u}_t = F\vec{y}_t$ can be regarded as a particular random output feedback with $\mathbf{\Pi} = F$.

- **An Equivalent Random Switching Output Feedback Model**
  - Introduce q mutually independent random sequences $\eta_r(t) \in \mathscr{D}_m$, $r \in \mathscr{D}_q$ that are i.i.d. with

$$\eta_r(t) \sim \mathrm{Col}_r(\mathbf{\Pi}), \quad r \in \mathscr{D}_q.$$

  - Then, the equivalent switching model for ROF $u_t \sim \mathbf{\Pi}\vec{y}_t$ is given by

$$\vec{u}_t = F_{\eta_t}\vec{y}_t$$

    with $F_{\eta_t} := [\vec{\eta}_1(t),\ \vec{\eta}_2(t),\ \cdots,\ \vec{\eta}_q(t)]$.
  - It is easily checked that $u_t \sim \mathbb{E}\vec{u}_t = \mathbb{E}(F_{\eta_t}\vec{y}_t) = (\mathbb{E}F_{\eta_t})\vec{y}_t = \mathbf{\Pi}\vec{y}_t$.

Assumption 1

The selection probability of u_t at each step t is completely determined by y_t; i.e., for any random event $\mathcal{E}$ satisfying

$$\mathbb{P}\{y_t = j, \mathcal{E}\} \neq \emptyset,$$

the following holds

$$\mathbb{P}\left\{u_t = i \;\middle|\; y_t = j, \mathcal{E}\right\} = \mathbb{P}\left\{u_t = i \;\middle|\; y_t = j\right\}.$$

4.1 Deterministic and Random Output Feedback

- **Closed-loop system under random output feedback**
  - The random output feedback is essentially a **time-invariant** strategy.
  - The closed-loop system under the random output feedback $u_t \sim \mathbf{\Pi}\vec{y}_t$ is a **homogeneous Markovian chain** with the 1-step transition probability matrix (TPM)

$$\mathbf{P}(\mathbf{\Pi}) = \mathbf{P} \ltimes (\mathbf{\Pi} H) \ltimes \mathbf{R}_{[n]} = \mathbf{P}(\mathbf{\Pi} H \otimes I_n)\mathbf{R}_{[n]},$$

where $\mathbf{R}_{[n]}$ is the power-reducing matrix and $\mathbf{P} = L \ltimes \boldsymbol{p}^{\omega}$.

(**See the next page for the derivation)**

Derivation of the closed-loop TPM:

$$\begin{aligned}\vec{x}_{t+1} &= L \ltimes \vec{w}_t \ltimes \vec{u}_t \ltimes \vec{x}_t \\ &= L \ltimes \vec{w}_t \ltimes F_{\eta_t} \ltimes H \ltimes \vec{x}_t \ltimes \vec{x}_t \\ &= L \ltimes \vec{w}_t \ltimes F_{\eta_t} \ltimes H \ltimes \mathbf{R}_{[n]} \ltimes \vec{x}_t\end{aligned}$$

$$\Downarrow$$

$$\mathbf{P}(\mathbf{\Pi}) = \mathbf{P} \ltimes (\mathbf{\Pi} H) \ltimes \mathbf{R}_{[n]}$$

Outline

4.2 Stabilizability by Random Output Feedback

- **Set of Output Feedback Gain Matrice:**

$$\mathcal{K} = \left\{ \mathbf{\Pi} \in \mathbb{R}^{m \times q} \,\middle|\, \mathbf{\Pi} \succeq 0,\ \mathbf{1}_m^\top \mathbf{\Pi} = \mathbf{1}_q^\top \right\}$$

4.2 Stabilizability by Random Output Feedback

- **Set of Output Feedback Gain Matrice:**

$$\mathcal{K} = \left\{ \mathbf{\Pi} \in \mathbb{R}^{m\times q} \;\middle|\; \mathbf{\Pi} \succeq 0,\ \mathbf{1}_m^\top \mathbf{\Pi} = \mathbf{1}_q^\top \right\}$$

- **Set of Equilibrium-preserving Output Feedback Gain Matrices:**

$$\mathcal{K}_{x_e} := \left\{ \mathbf{\Pi} \in \mathcal{K} \;\middle|\; \mathbf{P}(\mathbf{\Pi})\vec{x}_e = \vec{x}_e \right\}$$

4.2 Stabilizability by Random Output Feedback

- **Set of Output Feedback Gain Matrice:**

$$\mathcal{K} = \left\{ \mathbf{\Pi} \in \mathbb{R}^{m \times q} \;\middle|\; \mathbf{\Pi} \succeq 0,\ \mathbf{1}_m^\top \mathbf{\Pi} = \mathbf{1}_q^\top \right\}$$

- **Set of Equilibrium-preserving Output Feedback Gain Matrices:**

$$\mathcal{K}_{x_e} := \left\{ \mathbf{\Pi} \in \mathcal{K} \;\middle|\; \mathbf{P}(\mathbf{\Pi})\vec{x}_e = \vec{x}_e \right\}$$

- **Set of Stabilizing Output Feedback Gain Matrices:**

$$\mathcal{S}\mathcal{K}_{x_e} := \left\{ \mathbf{\Pi} \in \mathcal{K} \;\middle|\; \mathbf{\Pi} \text{ is asymptotically } x_e\text{-stabilizing} \right\}.$$

4.2 Stabilizability by Random Output Feedback

- **Set of Output Feedback Gain Matrice:**

$$\mathcal{K} = \left\{ \mathbf{\Pi} \in \mathbb{R}^{m \times q} \;\middle|\; \mathbf{\Pi} \succeq 0,\ \mathbf{1}_m^\top \mathbf{\Pi} = \mathbf{1}_q^\top \right\}$$

- **Set of Equilibrium-preserving Output Feedback Gain Matrices:**

$$\mathcal{K}_{x_e} := \left\{ \mathbf{\Pi} \in \mathcal{K} \;\middle|\; \mathbf{P}(\mathbf{\Pi})\vec{x}_e = \vec{x}_e \right\}$$

- **Set of Stabilizing Output Feedback Gain Matrices:**

$$\mathcal{SK}_{x_e} := \left\{ \mathbf{\Pi} \in \mathcal{K} \;\middle|\; \mathbf{\Pi} \text{ is asymptotically } x_e\text{-stabilizing} \right\}.$$

  - $\mathcal{SK}_{x_e} \subseteq \mathcal{K}_{x_e} \subseteq \mathcal{K}$
  - The system is asymptotically x_e-stabilizable iff there is a $\mathbf{\Pi} \in \mathcal{K}_{x_e}$ under which every state has a path to x_e in the closed-loop STG.

4.2 Stabilizability by Random Output Feedback

Proposition 4

Suppose that $\mathbf{\Pi}_1, \mathbf{\Pi}_2 \in \mathcal{K}_{x_e}$.

1. *If* $\mathbf{\Pi}_1 \sqsubseteq \mathbf{\Pi}_2$ *and* $\mathbf{\Pi}_1 \in \mathcal{SK}_{x_e}$. *Then,* $\mathbf{\Pi}_2 \in \mathcal{SK}_{x_e}$.
2. *If* $\mathbf{\Pi}_1 \sim_h \mathbf{\Pi}_2$, *then,* $\mathbf{\Pi}_1 \in \mathcal{SK}_{x_e}$ *iff* $\mathbf{\Pi}_2 \in \mathcal{SK}_{x_e}$.

Proof: (Claim 1) By Lemma 5, if $\mathbf{\Pi}_1 \sqsubseteq \mathbf{\Pi}_2$, then,

$$\mathbf{P}(\mathbf{\Pi}_1) = \mathbf{P} \ltimes (\mathbf{\Pi}_1 H) \ltimes \mathbf{R}_{[n]} \sqsubseteq \mathbf{P} \ltimes (\mathbf{\Pi}_2 H) \ltimes \mathbf{R}_{[n]} = \mathbf{P}(\mathbf{\Pi}_2).$$

The claims follow by using Corollary 16. □

- **A Partial Order Structure of $\mathcal{K}_{x_e}/\sim_h$**
  - **Equivalence Class:**

$$\langle \mathbf{\Pi} \rangle := \{\bar{\mathbf{\Pi}} \in \mathcal{K}_{x_e} \mid \bar{\mathbf{\Pi}} \sim_h \mathbf{\Pi}\}$$

  - **Quotient set:**

$$\mathcal{K}_{x_e}/\sim_h := \{\langle \mathbf{\Pi} \rangle \mid \mathbf{\Pi} \in \mathcal{K}_{x_e}\}$$

  - **Partial ordered set $(\mathcal{K}_{x_e}/\sim_h, \sqsubseteq)$:** If $\mathbf{\Pi}_1 \sqsubseteq \mathbf{\Pi}_2$, then,

$$\bar{\mathbf{\Pi}}_1 \sqsubseteq \bar{\mathbf{\Pi}}_2, \quad \forall \bar{\mathbf{\Pi}}_1 \in \langle \mathbf{\Pi}_1 \rangle, \forall \bar{\mathbf{\Pi}}_2 \in \langle \mathbf{\Pi}_2 \rangle.$$

In this case, we denote $\langle \mathbf{\Pi}_1 \rangle \sqsubseteq \langle \mathbf{\Pi}_2 \rangle$. Then, "$\sqsubseteq$" defines a partial order relation on $\mathcal{K}_{x_e}/\sim_h$:

    - ★ Reflexivity: $\langle \mathbf{\Pi} \rangle \sqsubseteq \langle \mathbf{\Pi} \rangle$ for any $\langle \mathbf{\Pi} \rangle \in \mathcal{K}_{x_e}/\sim_h$
    - ★ Antisymmetry: $\langle \mathbf{\Pi}_1 \rangle \sqsubseteq \langle \mathbf{\Pi}_2 \rangle$ and $\langle \mathbf{\Pi}_2 \rangle \sqsubseteq \langle \mathbf{\Pi}_1 \rangle$ implies $\langle \mathbf{\Pi}_1 \rangle = \langle \mathbf{\Pi}_2 \rangle$.
    - ★ Transitivity: $\langle \mathbf{\Pi}_1 \rangle \sqsubseteq \langle \mathbf{\Pi}_2 \rangle$ and $\langle \mathbf{\Pi}_2 \rangle \sqsubseteq \langle \mathbf{\Pi}_3 \rangle$ implies $\langle \mathbf{\Pi}_1 \rangle \sqsubseteq \langle \mathbf{\Pi}_3 \rangle$.

- **The unique maximum element of poset** $(\mathcal{K}_{x_e}/\sim_h, \sqsubseteq)$
  - **Uniformly distributed Feedback Gain Matrix**

$$\mathrm{Col}_j(\boldsymbol{\Gamma}_{x_e}) = \begin{cases} \frac{1}{m}\mathbf{1}_m, & j \neq h_{j_e} \\ \frac{1}{|\mathcal{U}_{x_e}|}\sum_{u\in\mathcal{U}_{x_e}} \vec{u}, & j = h_{j_e}, \end{cases} \qquad j \in [1:q]$$

    - ⋆ $\mathcal{U}_{x_e} := \{u \in \mathscr{D}_m \mid \mathbf{P} \ltimes \vec{u} \ltimes \vec{x}_e = \vec{x}_e\}$
    - ⋆ $h_{j_e} = \mathsf{idx}(H\vec{x}_e)$
  - $\langle \boldsymbol{\Gamma}_{x_e} \rangle$ **is the unique maximum element of poset** $(\mathcal{K}_{x_e}/\sim_h, \sqsubseteq)$
    1. $\boldsymbol{\Gamma}_{x_e} \in \mathcal{K}_{x_e}$.
    2. For any $\boldsymbol{\Pi} \in \mathcal{K}_{x_e}$, it holds that $\langle \boldsymbol{\Pi} \rangle \sqsubseteq \langle \boldsymbol{\Gamma}_{x_e} \rangle$.

Example 26

$$\begin{cases} \vec{x}_{t+1} = L \ltimes \vec{\omega}_t \ltimes \vec{u}_t \ltimes \vec{x}_t \\ \vec{y}_t = H\vec{x}_t \end{cases}$$

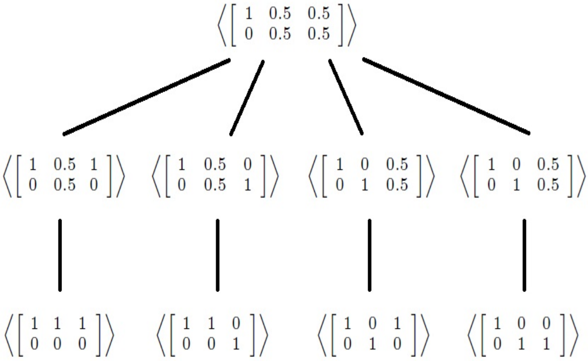
$$L_1 = \delta_4[1,2,3,3,2,2,3,4], \quad L_2 = \delta_4[2,1,3,1,2,3,4,2]$$

$$\omega_t \sim \boldsymbol{p}^{\omega} = [0.5,\ 0,5]^{\top}, \quad H = \delta_3[2,3,1,1], \quad x_e = 3$$

$$\mathbf{P} = L \ltimes \boldsymbol{p}^{w} = \left[\begin{array}{cccc|cccc} 0.5 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 1 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{array}\right]$$

$$j_e = \mathsf{idx}(H\vec{x}_e) = 1,\ \mathcal{U}_{x_e} = \{1\}$$

$$\mathcal{K}_{x_e} = \left\{ \mathbf{\Pi} = \begin{bmatrix} 1 & * & * \\ 0 & * & * \end{bmatrix} \right\}, \quad \mathbf{\Gamma}_{x_e} = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$



Hasse diagram of poset $(\mathcal{K}_{x_e}/\sim_h, \sqsubseteq)$

To check stabilizability, we only need to check whether $\mathbf{\Gamma}_{x_e}$ is stabilizing.

4.2 Stabilizability by Random Output Feedback

Theorem 27

A PLDS is asymptotically x_e-stabilizable by random output feedback iff x_e is a control-fixed point and $\mathbf{\Gamma}_{x_e}$ is asymptotically x_e-stabilizing.

- Every $\mathbf{\Pi} \in \langle \mathbf{\Gamma}_{x_e} \rangle$ can be a testing feedback gain matrix.
- This method is not valid for stabilizability under deterministic output feedback, because each equivalence class in $\mathscr{L}_{x_e}/\sim_h$ is a singleton. Thus, the maximal elements are not unique, where $\mathscr{L}_{x_e}$ denotes the set of equilibrium-preserving logical output feedback gain matrices.

Outline

- **The Problem of Designing Optimal Random Output Feedback:**
  - Quadratic cost function

$$J(x_0, \mathbf{\Pi}) := \sum_{t=0}^{\infty} \boldsymbol{e}_t^\top S \boldsymbol{e}_t = \sum_{t=0}^{\infty} [\boldsymbol{p}_t^x - \vec{x}_e]^\top S [\boldsymbol{p}_t^x - \vec{x}_e]$$

    where S is positive definite.
  - For any given initial output y_0, we aim to find a $\mathbf{\Pi} \in \mathcal{SF}_{x_e}$ to minimize

$$\mathcal{J}(y_0, \mathbf{\Pi}) := \max_{x_0 \in \mathcal{H}^{-1}(y_0)} J(x_0, \mathbf{\Pi}),$$

    where

$$\mathcal{H}^{-1}(y_0) := \{x_0 \in \mathscr{D}_n \mid H\vec{x}_0 = \vec{y}_0\}.$$

- **Zero Gap Between $\langle \mathbf{\Gamma}_{x_e} \rangle$ and $\mathcal{SK}_{x_e}$**

Proposition 5

If PLDS (6) is asymptotically x_e-stabilizable by random output feedback, then,

$$\langle \mathbf{\Gamma}_{x_e} \rangle \subseteq \mathcal{SK}_{x_e} \subseteq \overline{\langle \mathbf{\Gamma}_{x_e} \rangle} = \mathcal{K}_{x_e},$$

where $\overline{\langle \mathbf{\Gamma}_{x_e} \rangle}$ is the closure of $\langle \mathbf{\Gamma}_{x_e} \rangle$, that is,

$$\overline{\langle \mathbf{\Gamma}_{x_e} \rangle} := \left\{ \mathbf{\Pi} \in \mathcal{K} \;\middle|\; \exists \{\mathbf{\Gamma}_k\} \subseteq \langle \mathbf{\Gamma}_{x_e} \rangle, \text{ s.t. } \lim_{k \to \infty} \mathbf{\Gamma}_k = \mathbf{\Pi} \right\}.$$

Example 28 (Example 26 Revisited)

$$\mathcal{K}_{x_e} = \left\{ \mathbf{\Pi} = \begin{bmatrix} 1 & * & * \\ 0 & * & * \end{bmatrix} \right\}, \quad \mathbf{\Gamma}_{x_e} = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Example 28 (Example 26 Revisited)

$$\mathcal{K}_{x_e} = \left\{ \mathbf{\Pi} = \begin{bmatrix} 1 & * & * \\ 0 & * & * \end{bmatrix} \right\}, \quad \mathbf{\Gamma}_{x_e} = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Consider

$$\mathbf{\Pi} = \begin{bmatrix} 1 & 1 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \in \mathcal{K}_{x_e}$$

Obviously, $\mathbf{\Pi} \notin \langle \mathbf{\Gamma}_{x_e} \rangle$. We construct

$$\mathbf{\Pi}_k := \begin{bmatrix} 1 & 1 - 1/k & 0.5 \\ 0 & 1/k & 0.5 \end{bmatrix} \in \langle \mathbf{\Gamma}_{x_e} \rangle, \quad k = 1, 2, \cdots .$$

Then,

$$\lim_{k \to \infty} \mathbf{\Pi}_k = \mathbf{\Pi}.$$

Remark 2

- *It can be verified that* $\mathcal{J}(y_0, \mathbf{\Pi})$ *is continuous with respect to* $\mathbf{\Pi}$ *within* $\mathcal{SK}_{x_e}$. *Thus, by Proposition 5,*

$$\inf_{\mathbf{\Pi} \in \mathcal{SK}_{x_e}} \mathcal{J}(y_0, \mathbf{\Pi}) = \inf_{\mathbf{\Pi} \in \langle \mathbf{\Gamma}_{x_e} \rangle} \mathcal{J}(y_0, \mathbf{\Pi}) =: \lambda^*(y_0).$$

4.3 Optimal Random Output Feedback

- Based on the error system-based stability analysis in Section 2.3

$$\boldsymbol{e}_{t+1} = \mathbf{P}(\boldsymbol{\Pi})\boldsymbol{e}_t, \quad \boldsymbol{e}_0 \in \Delta_n - \vec{x}_e,$$

$$\Downarrow \quad \boldsymbol{e}_t = \mathbf{M}_{x_e}\boldsymbol{z}_1(t)$$

$$\boldsymbol{z}_1(t+1) = \underbrace{\mathbf{M}_{x_e}^{+}\mathbf{P}(\boldsymbol{\Pi})\mathbf{M}_{x_e}}_{\mathbf{D}(\boldsymbol{\Pi})}\boldsymbol{z}_1(t)$$

$$\mathbf{M}_{x_e} := [\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_{x_e-1}, \boldsymbol{\alpha}_{x_e+1}, \cdots, \boldsymbol{\alpha}_n].$$

$$\boldsymbol{\alpha}_i := \delta_n^i - \vec{x}_e, \quad i \in [1:n].$$

$$\Downarrow$$

$$J(x_0, \boldsymbol{\Pi}) = \sum_{t=0}^{\infty} \boldsymbol{e}_t^{\top} S \boldsymbol{e}_t = \sum_{t=0}^{\infty} \boldsymbol{z}_1^{\top}(t)\mathbf{M}_{x_e}^{\top} S \mathbf{M}_{x_e}\boldsymbol{z}_1(t)$$

Lemma 29

Suppose that $\mathbf{\Pi} \in \mathcal{SK}_{x_e}$. *The following claims hold:*

- *There exists an* $(n-1) \times (n-1)$ *positive-definite matrix* Ω *such that*

$$\mathbf{D}^\top(\mathbf{\Pi})\Omega\mathbf{D}(\mathbf{\Pi}) - \Omega = -\mathbf{M}_{x_e}^\top S \mathbf{M}_{x_e}$$

  and for any $i \in [1:n]$,

$$J(i, \mathbf{\Pi}) = \boldsymbol{\alpha}_i^\top (\mathbf{M}_{x_e}^+)^\top \Omega \mathbf{M}_{x_e}^+ \boldsymbol{\alpha}_i.$$

- *If an* $(n-1) \times (n-1)$ *positive-definite matrix* Ω *satisfies*

$$\mathbf{D}^\top(\mathbf{\Pi})\Omega\mathbf{D}(\mathbf{\Pi}) - \Omega \leq -\mathbf{M}_{x_e}^\top S \mathbf{M}_{x_e},$$

  then, for any $i \in [1:n]$, *it holds that*

$$J(i, \mathbf{\Pi}) \leq \boldsymbol{\alpha}_i^\top (\mathbf{M}_{x_e}^+)^\top \Omega \mathbf{M}_{x_e}^+ \boldsymbol{\alpha}_i.$$

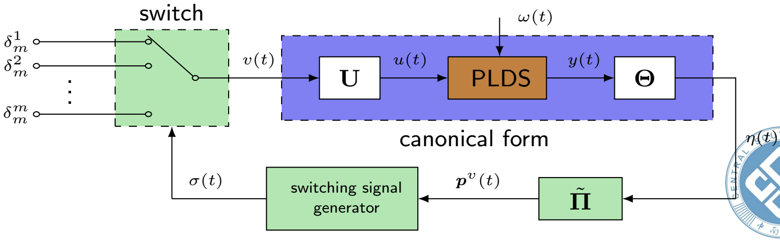
4.3 Optimal Random Output Feedback

- **Canonical form:** A PLDS satisfying

$$\mathcal{U}_{x_e} = \{1, 2, \cdots, r\}, \quad \vec{y}_e = H\vec{x}_e = \delta_q^1.$$

- If a PLDS is not of the canonical form, we can always convert it to the canonical form through the input and output transformations

$$\vec{u}_t = \mathbf{U}\vec{v}_t, \quad \vec{\eta}_t = \mathbf{\Theta}\vec{y}_t.$$



Theorem 30

Suppose that the PLDS is asymptotically x_e-stabilizable and denote $r := |\mathcal{U}_{x_e}|$. Then[a]

- *For a given $\lambda > 0$, if there exist symmetric matrices Q and Ω, a vector $\boldsymbol{\beta}$, and a matrix $\boldsymbol{\Sigma}$ such that*

$$\Omega > 0 \tag{7}$$

$$\begin{bmatrix} Q & -\mathbf{M}_{x_e}^{+}\mathbf{P}\left(\Gamma_{\boldsymbol{\beta},\boldsymbol{\Sigma}}\right)\mathbf{M}_{x_e} \\ * & \Omega - \mathbf{M}_{x_e}^{\top} S \mathbf{M}_{x_e} \end{bmatrix} > 0 \tag{8}$$

$$\boldsymbol{\alpha}_j^{\top}(\mathbf{M}_{x_e}^{+})^{\top}\Omega\mathbf{M}_{x_e}^{+}\boldsymbol{\alpha}_j < \lambda, \quad j \in \mathrm{idx}(\mathcal{H}^{-1}(y_0)) \setminus \{j_e\} \tag{9}$$

$$\boldsymbol{\beta} \succ 0, \quad \boldsymbol{\Sigma} \succ 0, \quad 1 - \mathbf{1}_{r-1}^{\top}\boldsymbol{\beta} > 0, \quad \mathbf{1}_{q-1}^{\top} - \mathbf{1}_{m-1}^{\top}\boldsymbol{\Sigma} \succ 0 \tag{10}$$

$$Q\Omega = I, \tag{11}$$

where

$$\Gamma_{\boldsymbol{\beta},\boldsymbol{\Sigma}} := \begin{bmatrix} 1 - \mathbf{1}_{r-1}^{\top}\boldsymbol{\beta} & \mathbf{1}_{q-1}^{\top} - \mathbf{1}_{m-1}^{\top}\boldsymbol{\Sigma} \\ I_{(m-1)\times(r-1)}\boldsymbol{\beta} & \boldsymbol{\Sigma} \end{bmatrix}.$$

Then, $\Gamma_{\boldsymbol{\beta},\boldsymbol{\Sigma}} \in \langle \boldsymbol{\Gamma}_{x_e} \rangle$ is asymptotically x_e-stabilizing and $\mathcal{J}(y_0, \Gamma_{\boldsymbol{\beta},\boldsymbol{\Sigma}}) < \lambda$.

- *For any $\lambda > \lambda^*(y_0)$, the LMIs (7), (8), (9), and (10) with equality constraint (11) have a solution.*

---

[a] Guo Yuqian et al. "Asymptotical Stabilization of Logic Dynamical Systems via Output-Based Random Control". In: *IEEE transactions on Automatic Control* 69.5 (2024), pp. 3286 –3293.

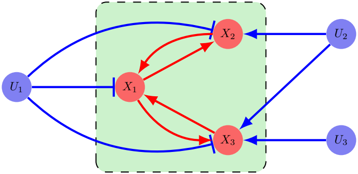
Remark 3

- *The LMIs with equality constraint can be transformed into the* **cone complementary problem** *which can be solved with the recursive algorithm proposed in [14].*
- *In addition, using the dichotomy for parameter λ, we can find an output feedback gain matrix $\mathbf{\Pi} \in \langle \mathbf{\Gamma}_{x_e} \rangle$ such that the cost $\mathcal{J}(y_0, \mathbf{\Pi})$ approximates the optimal value $\lambda^*(y_0)$ with any given accuracy.*

[14] L. E. Ghaoui and F. Oustry, A cone complementarity linearization algorithm for static output-feedback and related problems, *IEEE Transactions on Automatic Control*, vol. 42, no. 8, pp. 1171–1176, 1997.

4.3 Optimal Random Output Feedback

- **Reduced model for the lac operon in the bacterium *Escherichia coli*[15]**



Normal and blunt arrows indicate positive and negative interactions, respectively.

$$
\begin{cases}
X_1(t+1) = \neg U_1(t) \wedge (X_2(t) \vee X_3(t)) \\
X_2(t+1) = \neg U_1(t) \wedge U_2(t) \wedge X_1(t) \\
X_3(t+1) = \neg U_1(t) \wedge (U_2(t) \vee (U_3(t) \wedge X_1(t)))
\end{cases}
$$

- We consider the problem of stabilizing $X_e = (1, 0, 1)$, which represents the ON status of the *lac* perion [11], and minimizing

$$
\mathcal{J}(y_0, \mathbf{\Pi}) := \max_{x_0 \in \mathcal{H}^{-1}(y_0)} \sum_{t=0}^{\infty} \|\boldsymbol{e_p}(t)\|^2
$$

[15] A. Veliz-Cuba and B. Stigler, Boolean models can explain bistability in the *lac* operon. *Journal of Computational Biology*, vol. 18, no. 6, pp. 783–794, 2011.

4.3 Optimal Random Output Feedback

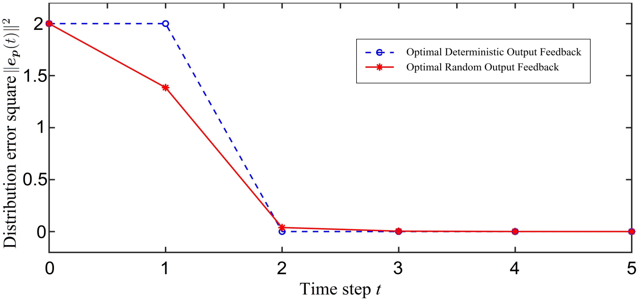
- **Comparison between TIDOF and random output feedback**

measurable states	initial output	minimum cost under TIDOF	minimum cost under random output feedback
x_1	δ_2^1 δ_2^2	4.00 4.00	4.00 3.89
x_1, x_2	δ_4^1, δ_4^3 δ_4^2 δ_4^3	2.00 4.00 4.00	2.00 4.00 3.89
x_1, x_3	δ_4^1, δ_4^3 δ_4^2 δ_4^4	2.00 4.00 4.00	2.00 3.43 3.74
x_2, x_3	δ_4^1 δ_4^2 δ_4^3 δ_4^4	/ / / /	2.95 2.99 6.55 8.00
x_1, x_2, x_3	$\delta_8^1, \delta_8^2, \delta_8^5, \delta_8^6, \delta_8^7$ δ_8^4 δ_8^8	2.00 4.00 4.00	2.00 3.43 3.74

- **Time-domain Simulation:**
  - Measurable states are $y_1 = x_1$ and $y_2 = x_3$, $H = \delta_4[1,2,1,2,3,4,3,4]$.
  - Initial output is $y_0 = \delta_4^2$, $\mathcal{H}^{-1}(y_0) = \{\delta_8^2, \delta_8^4\}$.
  - The optimal deterministic and random output feedback gain matrices:

$$F^* = \delta_8[7,7,6,6], \quad \mathbf{\Pi}^* = \begin{bmatrix} 0 & 0.0297 & 0.0016 & 0.0408 \\ 0 & 0.0297 & 0.0016 & 0.0408 \\ 0 & 0.0297 & 0.0016 & 0.0408 \\ 0 & 0.0297 & 0.0016 & 0.0408 \\ 0 & 0.2072 & 0.4949 & 0.3723 \\ 0 & 0.2072 & 0.4949 & 0.3723 \\ 1 & 0.4329 & 0.0020 & 0.0462 \\ 0 & 0.0339 & 0.0020 & 0.0462 \end{bmatrix}.$$

4.3 Optimal Random Output Feedback



The curves of $\|\boldsymbol{e_p}(t)\|^2$ with the initial state $x_0 = \delta_8^4$

Conclusion

- Basic theories of stability and feedback stabilization for PLDSs were reviewed.
- New stability result and the random output feedback for PLDSs were discussed.

Thank you!