# Decoupling of Boolean Control Networks

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# Outline

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# Background

- Decoupling refers to the separation of two or more originally interrelated systems or modules to minimise their mutual influence.
- In control systems, decoupling usually refers to the elimination of cross-coupling between inputs and outputs by designing appropriate control strategies so that each output of a multiple-input multiple-output (MIMO) system is controlled by only one of the corresponding inputs, and at the same time, each input can control only one output.
- Disturbance Decoupling; Decomposition. (Decoupling control refers to the design of controls to reduce or eliminate the interactions between variables in a multivariable system in order to achieve better control performance)

# Background

The decoupling problem is an important topic in control theory and system design both for linear and nonlinear systems [1,2]. Decoupling has a wide range of applications in several fields, including but not limited to:

- Industrial automation: decoupling control is used to improve the stability and efficiency of the production process in chemical, metallurgical, and electric power industries.
- Robot control: Through decoupling control, independent control of each joint of the robot can be realised to improve the motion precision and flexibility of the robot.
- Aerospace: In the attitude control of aircraft, decoupling control is used to reduce the mutual influence between different control channels and improve the stability and safety of the aircraft.
- Gene regulatory networks: In actual gene regulatory networks, coupling is prevalent, which makes it difficult to understand and control the evolution of gene regulatory networks.

[1] W. Wonham, Linear Multivariable Control: A Geometric Approach, Berlin, Springer-Verlag, 1979.
[2] A. Isidor, Nonlinear Control Systems, Berlin, Springer-Verlag, 1995.

# Outline

# Algebraic form of BNs

## Definition 1.1,[1]

Set $A = (a_{ij}) \in \mathcal{R}_{m \times n}$, $B = (b_{ij}) \in \mathcal{R}_{p \times q}$. Let $\alpha =$lcm$(n, p)$ be the least common multiple of $n$ and $p$. The semi-tensor product of $A$ and $B$ is defined as

$$A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}}). \tag{1}$$

## Remark

*In Definition 1, when $n = p$, the semi-tensor product becomes the conventional matrix product. Hence the semi-tensor product is a generalization of the traditional matrix product. In the following discussion, $A \ltimes B$ is denoted by $AB$.*

[1] D. Cheng, H. Qi and Z. Li, Analysis and control of Boolean networks: a semi-tensor product approach. London: Springer, 2011.

# Algebraic form of BNs

## Lemma 1.1,[1]

For a logical function $f(X_1, X_2, \cdots, X_n) : \mathcal{B}^n \mapsto \mathcal{B}$, there is a unique matrix $M_f \in \mathcal{L}_{2^n \times 2^n}$, called the structure matrix of $f$, such that

$$f(x_1, x_2, \cdots, x_n) = M_f \ltimes_{i=1}^n x_i,$$

where $X_i \sim x_i$, $i = 1, 2, \cdots, n$.

Consider the BCN described as follows:

$$\begin{array}{rcl} X(t+1) & = & f(X(t), U(t)), \\ Y(t) & = & h(X(t)), \end{array} \tag{2}$$

where $X = [X_1, X_2, \ldots, X_n]^{\mathrm{T}} \in \mathcal{B}^n$, $U = [U_1, U_2, \ldots, U_m]^{\mathrm{T}} \in \mathcal{B}^m$ , $Y = [Y_1, Y_2, \ldots, Y_p]^{\mathrm{T}} \in \mathcal{B}^p$ are the state vector, the input vector and the output vector respectively, $\mathcal{B} = \{0, 1\}$.

# Algebraic form of BNs

System (2) can be converted into the following algebraic form

$$\begin{aligned} x(t+1) &= Lu(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{3}$$

where $x \in \Delta_{2^n}$, $y \in \Delta_{2^p}$, $L \in \mathcal{L}_{2^n \times 2^{m+n}}$ and $H \in \mathcal{L}_{2^p \times 2^n}$.

# State spaces of BNs

Consider the logical mapping $g : \mathcal{B}^n \to \mathcal{B}^n$ defined by

$$g : (X_1, X_2, \cdots, X_n) \mapsto (Z_1, Z_2, \cdots, Z_n). \tag{4}$$

$g$ is called a logical coordinate transformation if it is a bijection [1].

Let $z = Tx$ be the algebraic form of the logical coordinate transformation (4), where $T \in \mathcal{L}_{2^n \times 2^n}$ is the structure matrix of $g$, $x = \ltimes_{i=1}^n x_i$, $z = \ltimes_{i=1}^n z_i$, $X_i \sim x_i$, $Z_i \sim z_i$. Then, $g$ is a logical coordinate transformation if and only if $T$ is a nonsingular logical matrix, i.e., a permutation matrix [1].

[1] D. Cheng, H. Qi and Z. Li, Analysis and control of Boolean networks: a semi-tensor product approach. London: Springer, 2011.

# State spaces of BNs

## Definition 1.2 [1]

- Consider a BN. The *state space* $\mathcal{X}$ is defined as the set of all logical functions of $\{X_1, X_2, \cdots, X_n\}$, denoted by $\mathcal{F}_l\{X_1, X_2, \cdots, X_n\}$.
- Let $Z_1, Z_2, \cdots, Z_r \in \mathcal{X}$. The subspace generated by $\{Z_1, Z_2, \cdots, Z_r\}$ is defined as the set of logical functions of $\{Z_1, Z_2, \cdots, Z_r\}$, denoted by $\mathcal{Z} = \mathcal{F}_l\{Z_1, Z_2, \cdots, Z_r\}$.
- A subspace $\mathcal{Z} = \mathcal{F}_l\{Z_1, Z_2, \cdots, Z_k\} \subset \mathcal{X}$ is called a regular subspace of the dimension $k$, if there are $Z_{k+1}, z_{k+2}, \cdots, Z_n \in \mathcal{X}$, such that $g : (X_1, X_2, \cdots, X_n) \mapsto (Z_1, Z_2, \cdots, Z_n)$ is a logical coordinate transformation.

## Proposition 1.1 [1]

Let $\mathcal{Z} = \mathcal{F}_l\{Z_1, Z_2, \cdots, Z_r\} \subset \mathcal{X}$ and its algebraic form is $z = Mx$, $M \in \mathcal{L}_{2^r \times 2^n}$. Then $\mathcal{Z}$ is a regular subspace iff $M\mathbf{1}_{2^n} = 2^{n-r}\mathbf{1}_{2^r}$.

[1] Cheng, D., Qi, H. State-space analysis of Boolean networks. IEEE Transactions on Neural Networks, 21(4), 584-594, 2010.

[2] Cheng D, Li Z, and Qi H, Realization of Boolean control networks, Automatica, 46(1): 62-69, 2010.

# State spaces of BNs [1-3]

## Definition 1.3

- A regular subspace $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \cdots, Z_k\} \subset \mathcal{X}$ is called an invariant subspace of the dimension $k$, if there are $Z_{k+1}, z_{k+2}, \cdots, Z_n \in \mathcal{X}$, such that $g : (X_1, X_2, \cdots, X_n) \mapsto (Z_1, Z_2, \cdots, Z_n)$ is a logical coordinate transformation and under $Z$ the system can be expressed as

$$\begin{array}{rcl} Z^1(t+1) & = & f_1(Z^1(t)), \\ Z^2(t+1) & = & f_2(Z(t)), \end{array} \tag{5}$$

  where $Z^1 = \{Z_1, Z_2, \cdots, Z_k\}$, $Z^2 = \{Z_{k+1}, z_{k+2}, \cdots, Z_n\}$.
- Let $Y = \{Y_1, Y_2, \cdots, Y_p\} \in \mathcal{X}$. A regular subspace $\mathcal{Z} \subset \mathcal{X}$ is called a $Y$-friendly subspace if $Y_i \in \mathcal{Z}$, $i = 1, 2, \cdots, p$. A $Y$-friendly subspace is called a minimal $Y$-friendly subspace if there is no $Y$-friendly subspace of smaller dimension than it.

[1] Cheng, D., Qi, H. State-space analysis of Boolean networks. IEEE Transactions on Neural Networks, 21(4), 584-594, 2010.
[28] Cheng D, Li Z, and Qi H, Realization of Boolean control networks, Automatica, 46(1): 62-69, 2010.

# Vertex partitions

## Definition 1.4

For a digraph $\mathcal{G} = (V, \mathcal{E})$, let $P_l$, $l = 1, 2, \cdots, \mu$ be some subsets of $V$.

(i) $\{P_l\}_{l=1}^{\mu}$ is called a *vertex partition* of $\mathcal{G}$, if $\cup_{l=1}^{\mu} P_l = V$ and $P_i \cap P_j = \emptyset$ for any $i \neq j$.

(ii) A vertex partition $\{P_l\}_{l=1}^{\mu}$ is called an *equal vertex partition* (E-VP) if $|P_l| = |V|/\mu$ for every $l = 1, 2, \cdots, \mu$.

(iii) A vertex partition $\{P_l\}_{l=1}^{\mu}$ is called a *perfect vertex partition* (P-VP) if for each $l$ there exists $\alpha_l$ such that $\mathcal{N}(P_l) \subset P_{\alpha_l}$.

(iv) A vertex partition $\{P_l\}_{l=1}^{\mu}$ is called a *concolorous vertex partition* (C-VP) if for any $l \in \{1, 2, \cdots, \mu\}$, all the vertices in $P_l$ have the same color.

(v) CP-VP; CPE-VP.

# Vertex-colored State Transition Graph

Consider BCN

$$\begin{aligned} x(t+1) &= [L_1, L_2, \cdots, L_{2^m}]u(t)x(t), \\ y(t) &= Hx(t). \end{aligned}$$

- $B = (b_{ij}) = \sum_{k=1}^{2^m} {}_{\mathcal{B}} L_k$ is a adjacency matrix of the (STG) $\mathcal{G}$ of the BCN.
- Use the output function to assign a color for every vertex of $\mathcal{G}$ in such way that $\mu$ and $\lambda$ are of the same color if and only if

$$H\delta_{2^n}^{\mu} = H\delta_{2^n}^{\lambda},$$

  then the constructed STG $\mathcal{G}$ is called the *vertex-colored STG* associated with $H$ and $B$.
- $\mathcal{G}_i$, denote the vertex-colored STG associated with $H$ and $L_i$.
- $\mathcal{G} = \cup_{i=1}^{2^m} \mathcal{G}_i$.

# Example 1.1

Consider a BCN with

$$H = \delta_2[1\ 1\ 2\ 2\ 1\ 2\ 2\ 2],$$

$$L = [L_1, L_2] = \delta_{2^3}[3\ 4\ 4\ 6\ 7\ 7\ 8\ 7\ 2\ 5\ 1\ 2\ 2\ 5\ 5\ 5].$$

Let $y = \delta_2^1, \delta_2^2$, represent gray and white, respectively. Then based on $L_1$, $L_2$, $L$ and $H$, the vertex-colored STG $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}$ can be obtained shown in Fig. 1.

$$\{S_1 = \{\delta_8^1, \delta_8^2, \delta_8^5\}, S_2 = \{\delta_8^3, \delta_8^4, \delta_8^6, \delta_8^7, \delta_8^8\}\}.$$

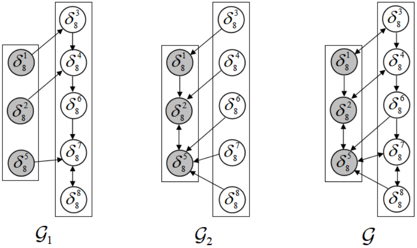

Figure 1: CCP-VP

# Outline

# DD dependent on decomposition

Consider the BCN with disturbances

$$\begin{array}{rcl} x(t+1) & = & Fu(t)\xi(t)x(t), \\ y(t) & = & Hx(t), \end{array} \tag{6}$$

where $x \in \Delta_{2^n}$, $y \in \Delta_{2^p}$, $F \in \mathcal{L}_{2^n \times 2^{n+q+m}}$ and $H \in \mathcal{L}_{2^p \times 2^n}$.

## Definition 2.1[1]

Consider BCN (6). The DDP is solvable if we can find a feedback control

$$u(t) = Kx(t) \tag{7}$$

and a logical coordinate transformation $z = Tx$ such that under $z$ coordinate frame the closed-loop system becomes

$$\begin{array}{rcl} z^1(t+1) & = & G_1 z^1(t), \\ z^2(t+1) & = & G_2\xi(t)z(t), \\ y(t) & = & Ez^1(t), \end{array} \tag{8}$$

where $z = z^1 z^2$.

[1] D. Cheng, Disturbance decoupling of Boolean control networks, IEEE Trans. Automat. Control, 56 (1), pp. 2-10, 2011.

The basic idea proposed [1] is dividing into two steps:

- Step 1, finding a $Y$-friendly subspace

$$\begin{aligned} z^1(t+1) &= F_1 u(t) z^1(t) z^2(t) \xi(t), \\ z^2(t+1) &= F_2 u(t) z^1(t) z^2(t) \xi(t), \\ y(t) &= G z^1(t), \end{aligned} \tag{9}$$

- Step 2, designing a control, such that the complement coordinate sub-basis $z^2$ and the disturbances $\xi$ can be deleted from the dynamics of $z^1$.

$$\begin{aligned} z^1(t+1) &= F z^1(t), \\ y(t) &= G z^1(t), \end{aligned} \tag{10}$$

[1] D. Cheng, Disturbance decoupling of Boolean control networks, IEEE Trans. Automat. Control, 56 (1), pp. 2-10, 2011.

# Find a $Y$-friendly subspace $z^1$

$$\begin{aligned} z^1(t) &= T_0 x(t), \\ y(t) &= G z^1(t). \end{aligned} \quad (11)$$

$$y(t) = Gz^1(t) = GT_0x(t) = Hx(t).$$

**例 7.1.2** 设 $\mathcal{X} = \mathcal{F}_\ell\{X_1, X_2, X_3\}$,

$$Y = (X_1 \wedge X_2 \wedge X_3) \vee (X_1 \wedge \neg X_2) \vee (\neg X_1 \wedge X_2 \wedge X_3) \vee (\neg X_1 \wedge \neg X_2),$$

求一个 $Y$ 友好子空间.

容易计算 $Y$ 的结构矩阵 $H$ 为

$$H = \delta_2[1, 2, 1, 1, 2, 1, 1, 1].$$

于是 $n_1 = 6$, $n_2 = 2$, $2 = 2^{n-r}$ 是 $(n_1,\ n_2)$ 的 2 型公因子, 于是 $r = 2$, $m_1 = 3$, $m_2 = 1$, 并且,

$$J^1 = \{1, 3, 4, 6, 7, 8\} := \{J_1^1, J_2^1, J_3^1\},$$
$$J^2 = \{2, 5\} := \{J_1^2\}.$$

于是

$$\mathrm{Col}_i(T_0) = \begin{cases} \delta_4^1, & i = 1, 3, \\ \delta_4^2, & i = 4, 6, \\ \delta_4^3, & i = 7, 8, \\ \delta_4^4, & i = 2, 5. \end{cases}$$

$$T_0 = \delta_4[1, 4, 1, 2, 4, 2, 3, 3].$$

$$G = \delta_2[1,1,1,2].$$

为得到新坐标 $Z$, 令

$$z_3 = M_3 x = \delta_2[1,1,2,1,2,2,1,2]x,$$

则坐标变换为

$$T = T_0 * M_3 = \delta_8[1,7,2,3,8,4,5,6].$$

于是有

$$z_1 = \delta_2[1,2,1,1,2,1,2,2]x,$$
$$z_2 = \delta_2[1,2,1,2,2,2,1,1]x,$$
$$z_3 = \delta_2[1,1,2,1,2,2,1,2]x.$$

而输出为

$$Y = Z_1 = Z_1 \vee (\neg Z_1 \wedge Z_2).$$

**算法 7.1.1** 第一步: 设输出 $Y=\{Y_1,Y_2,\cdots,Y_p\}$, $y=\ltimes_{i=1}^{p}y_i$ 为其向量表示, 计算 $Y$ 的结构矩阵

$$y=Hx:=\delta_{2^p}[i_1,i_2,\cdots,i_{2^n}]x. \tag{7.1.16}$$

第二步: 计算

$$n_i=\left|\{j\mid \mathrm{Col}_j(H)=\delta_{2^p}^i\}\right|,\quad i=1,2,\cdots,2^p.$$

第三步: 找出 $n_i, i=1,2,\cdots,2^p$ 的 (最大) 2 型公因子 $2^{n-r}$, 记 $n_i=m_i2^{n-r}$. 定义

$$\begin{aligned}J^s&=\{j\mid \mathrm{Col}_j(H)=\delta_{2^p}^s\}\\&=\{\alpha_1,\alpha_2,\cdots,\alpha_{n_s}\}\\&:=\{J_1^s,J_2^s,\cdots,J_{m_s}^s\},\quad s=1,2,\cdots,2^p.\end{aligned}$$

(这里, $|J_j^s|=2^{n-r}$, $s=1,2,\cdots,2^p$, $j=1,2,\cdots,m_s$.)

第四步: 构造 $2^r$ 维 $Y$ 友好子空间 $z=T_0x$, 通过构造其结构矩阵 $T_0$:

$$\mathrm{Col}_i(T_0)=\begin{cases}\delta_{2^r}^{j}, & i\in J_j^1,\\ \delta_{2^r}^{m_1+j}, & i\in J_j^2,\\ \qquad\cdots\cdots & \\ \delta_{2^r}^{m_1+m_2+\cdots+m_{2^p-1}+j}, & i\in J_j^{2^p}.\end{cases} \tag{7.1.17}$$

同时可得 $y=Qz$, 这里,

$$Q=\delta_{2^p}\left[\underbrace{1,\cdots,1}_{m_12^{n-c}},\underbrace{2,\cdots,2}_{m_22^{n-c}},\cdots,\underbrace{2^p,\cdots,2^p}_{m_{2^p}2^{n-c}}\right]. \tag{7.1.18}$$

# Controller Design

## Definition 2.2 [1]

Let

$$G = (g_1, ..., g_s) : \mathcal{B}^n \mapsto \mathcal{B}^s \tag{12}$$

be a logical mapping. The variable $x_i$ is said to be redundant if

$$g_j(x_1, ..., x_{i-1}, 1, x_{i-1}, ..., x_n) = g_i(x_1, ..., x_{i-1}, 0, x_{i-1}, ..., x_n)$$

for $j = 1, 2, \cdots, s$.

## Lemma 2.2 [1]

Let $M_G \in \mathcal{L}_{2^s \times 2^n}$ be the structure matrix of the logical mapping (12) and let an integer $r \leq n$ be given. Split $M_G$ into $2^r$ blocks as

$$M_G = [M_1, M_2, \cdots, M_{2^r}],$$

Then $x_{r+1}, \cdots, x_n$ are all redundant variables iff $rankM_i = 1, i = 1, 2, \cdots, 2^r$.

[1] M. Yang, R. Li, T. Chu, Controller design for disturbance decoupling of Boolean control networks, Automatica, 49 (1), pp. 273-277, 2013.

# Controller Design

Suppose that $z^1$ is a $r$-dimensional $Y$-friendly subspace and under $z$

$$z^1(t+1) = Lu(t)z^1(t)z^2(t)\xi(t). \tag{13}$$

The aim of DDP is to design state feedback controls such that $z^1$ is an invariant space. That is, $z^2$ and $\xi$ are redundant variables of (13).

Split $L$ into $2^m$ equal blocks as

$$L = [L_1 \ L_2 \cdots \ L_{2^m}],$$

and then split each $L_i$ into $2^n$ equal block as

$$L_i = [L_{i,1} \ L_{i,2} \cdots \ L_{i,2^n}].$$

For each $j \in [1, 2^n]$, define

$$\Lambda_j = \{k : \exists i \in [1, 2^m] \text{ such that } Col(L_{i,j}) = \{\delta_{2^r}^k\}\}.$$

and let

$$E_j^k = \{i : Col(L_{i,j}) = \{\delta_{2^r}^k\}\}.$$

For any $j \in [1, 2^r]$, denote

$$\Pi_j = \cap_{l=1}^{2^{n-r}} \Lambda_{(j-1)2^{n-r}+l}.$$

## Theorem 2.1 [1]

The DDP is solvable iff there exists a $Y$-friendly subspace $Z^1$ such that $\Pi_j \neq \emptyset$ for each $j \in [1, 2^r]$.

## Theorem 2.2 [1]

If the DDP is solvable, then the corresponding feedback control matrix is $K = \delta_{2^m}[v_1, v_2, \cdots, v_{2^n}]$, where $v_s \in E_s^{k_j}$, and $k_j \in \Pi_j$.

[1] M. Yang, R. Li, T. Chu, Controller design for disturbance decoupling of Boolean control networks, Automatica, 49 (1), pp. 273-277, 2013.

## An algorithm for solving DDP dependent on decomposition

- Step 1: Find a $Y$-friendly subspace $z^1$.
- Step 2: Check the conditions in Theorem 2.1.
- Step 3: If the DDP is solvable, then using Theorem 2.2 design controllers.
- Step 4: If the DDP is unsolvable, then go to Step 1 to find another $Y$-friendly subspace $\tilde{z}^1$.

[1] S. Wang and H. Li, New Results on the Disturbance Decoupling of Boolean Control Networks, IEEE Control Systems Letters, vol. 5, no. 4, pp. 1157-1162, 2021

# Some work on the DDP of BNs

The idea has been successfully applied to solve the DDP of mix-valued logical networks, switched BCNs and singular BNs.


- D. Cheng, Disturbance decoupling of Boolean control networks, IEEE Trans. Automat. Control., 56 (1), pp. 2-10, 2011.
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- S. Wang and H. Li, New Results on the Disturbance Decoupling of Boolean Control Networks, IEEE Control Systems Letters, vol. 5, no. 4, pp. 1157-1162, 2021
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- M. Yang, R. Li, T. Chu, Controller design for disturbance decoupling of Boolean control networks, Automatica, 49 (1), pp. 273-277, 2013.

# Some work on the DDP of BNs


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- L. Zhang, J. Feng, and X. Feng, Further results on disturbance decoupling of mix-valued logical networks, IEEE Trans. Autom. Control, 59 (6), pp. 1630-1634, 2014.
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# Some work on the DDP of BNs

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- K. Sarda, A. Yerudkar, C. Vecchio, Disturbance decoupling control design for Boolean control networks: a Boolean algebra approach, IET Control Theory Applications, 14(16):2339-2347, 2020.
- R. Zhao, J. Feng, B. Wang, R. Leone, Disturbance decoupling of Boolean networks via robust indistinguishability method, Applied Mathematics and Computation 457, 128220, 2023.

Consider the BN with disturbances

$$
\begin{aligned}
x(t+1) &= F\xi(t)x(t), \\
y(t) &= Hx(t),
\end{aligned} \tag{14}
$$

where $x \in \Delta_{2^n}$, $y \in \Delta_{2^p}$, $F \in \mathcal{L}_{2^n \times 2^{n+q}}$ and $H \in \mathcal{L}_{2^p \times 2^n}$.

## Definition 2.1 [1]

Consider BN (14). The DDP is solvable if there exists a logical coordinate transformation $z = Tx$ such that under $z$ coordinate frame the system becomes

$$
\begin{aligned}
z^{[1]}(t+1) &= G_1\xi(t)z(t), \\
z^{[2]}(t+1) &= G_2 z^{[2]}(t), \\
y(t) &= Ez^{[2]}(t),
\end{aligned} \tag{15}
$$

where $z = z^{[1]}z^{[2]}$.

[1] D. Cheng, Disturbance decoupling of Boolean control networks, IEEE Trans. Automat. Control, 56 (1), pp. 2-10, 2011.

- In the traditional control theory, the DDP is independent of any system decomposition [1].

- For BNs, whether the system decomposition is necessary to ensure that the output $y$ is unaffected by the disturbance $\xi$ ?

[1] W. Wonham, Linear Multivariable Control: A Geometric Aproach,2nd ed. Springer, Berlin, 1979.

## Example

Consider a BN with algebraic form

$$\begin{aligned} x(t+1) &= F\xi(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{16}$$

where $F = \delta_4[2,3,2,3,4,4,2,2]$, $H = \delta_2[1,2,2,2]$.

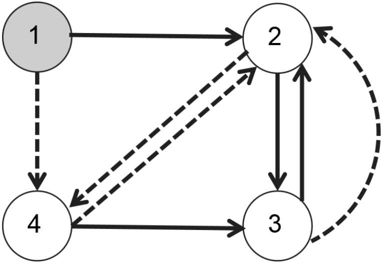



Figure 2: The vertex-colored state transition graph of system (16)

# DD independent on decomposition

## Definition 2.3 ODD [1]

Consider a BN with disturbances. The original disturbance decoupling is said to be implemented if, for each initial state $X(0) \in \mathcal{D}^n$, the output sequence $\{Y(s)\}_{s=0}^{+\infty}$ is the same for every disturbance sequence $\{\xi(s)\}_{s=0}^{+\infty}$ with each $\xi(s) \in \mathcal{D}^n$.

- This definition does not address the system decomposition, which is more general.

[1] Y. Li, J. Zhu, On disturbance decoupling problem of Boolean control networks, Asian J. Control, 21, pp. 2543-2550, 2019.

# Algebraic conditions for the ODD of BNs

## Theorem 2.3 [1]

For the BN

$$\begin{aligned} x(t+1) &= F\xi(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{17}$$

the ODD is implemented, i.e., the output vector $y(t)$ is unaffected by the disturbances for any $t$ and any initial state $x(0)$, if and only if

$$\begin{cases} P_1 := HF_1 = HF_2 = \cdots = HF_{2^q}, \\ P_2 := P_1 F_1 = P_1 F_2 = \cdots = P_1 F_{2^q}, \\ \quad \vdots \\ P_{t+1} := P_t F_1 = P_t F_2 = \cdots = P_t F_{2^q}, \\ \quad \vdots \end{cases} \tag{18}$$

where $F = [F_1, F_2, \cdots, F_{2^q}]$.

# Algebraic conditions for ODD of BNs

Assume that the output vector $y(t)$ is unaffected by the disturbances. A straightforward computation shows that

$$y(1) = Hx(1) = HF\xi(0)x(0) = HF_i x(0)$$

as $\xi(0) = \delta_{2^q}^i$, $i = 1, 2, \ldots, 2^q$. Since $y(1)$ is undisturbed, we have

$$HF_1 = HF_2 = \cdots = HF_{2^q} =: P_1.$$

Furthermore, we have

$$y(2) = Hx(2) = HF\xi(1)F\xi(0)x(0) = P_1 F_i x(0)$$

as the disturbance vector $\xi(0) = \delta_{2^q}^i$, $i = 1, 2, \ldots, 2^q$. Since $y(2)$ is undisturbed, we have

$$P_1 F_1 = P_1 F_2 = \cdots = P_1 F_{2^q} =: P_2.$$

Repeating this argument yields

$$y(t+1) = HF\xi(t)F\xi(t-1)\cdots F\xi(0)x(0) = P_tF_ix(0)$$

as the disturbance vector $\xi(0) = \delta_{2^q}^i$, $i = 1, 2, \ldots, 2^q$. Since the $y(t+1)$ is undisturbed, we have

$$P_tF_1 = P_tF_2 = \cdots = P_tF_{2^q} =: P_{t+1}$$

for every $t \geq 1$. So all the equalities in (18) hold.

Conversely, from (18), it follows that

$$y(t) = HF\xi(t-1)\cdots F\xi(0)x(0) = P_tx(0), \ \forall\ t \geq 1,$$

which implies that $y(t)$ is undisturbed for any $t$ and any initial state $x(0)$.

# Algebraic conditions for the ODD of BNs

## Corollary 2.1

The ODD of the BN is implemented if and only if

$$\begin{cases} P_1 := HF_1 = HF_2 = \cdots = HF_{2^q}, \\ P_2 := P_1 F_1 = P_1 F_2 = \cdots = P_1 F_{2^q}, \\ \quad \vdots \\ P_\mu := P_{\mu-1} F_1 = P_{\mu-1} F_2 = \cdots = P_{\mu-1} F_{2^q}, \end{cases} \tag{19}$$

where $\mu$ is some positive integer.

## Theorem 2.3;[1]

The ODD of the BN is implemented if and only if

$$HB^i \in \mathcal{L}_{2^p \times 2^n}, \quad i = 1, 2, \ldots, \mu, \tag{20}$$

where $\mu$ is some positive integer and $B = \sum_{i=1}^{2^q} {}_{\mathcal{B}} F_i$.

# Partition conditions

## Definition 2.4

A vertex partition $\mathcal{S} = \{S_l\}_{l=1}^{\mu}$ of $V$ is called a *concolorous perfect vertex partition (CP-VP)* if

(i) for any $l \in \{1, 2, \cdots, \mu\}$, all the vertices in $S_l$ have the same color (output),

(ii) for any $l \in \{1, 2, \cdots, \mu\}$, there exists an $\alpha_l$ such that $\mathcal{N}(S_l) \subset S_{\alpha_l}$.

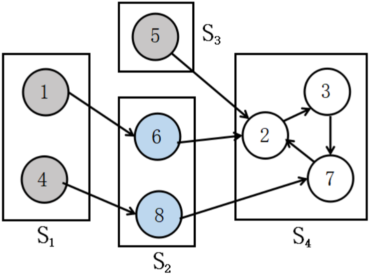



Figure 3: CP-VP

# Partition conditions

## Definition 2.5

A vertex partition $\mathcal{S} = \{S_l\}_{l=1}^{\mu}$ of $V$ is called a *concolorous perfect equal vertex partition (CPE-VP)* if $\mathcal{S} = \{S_l\}_{l=1}^{\mu}$ of $V$ is a CP-VP and an equal vertex partition.

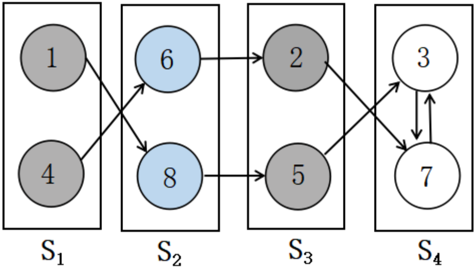



Figure 4: CPE-VP

# Partition condition for the DD of BNs

## Remark

*From the definition of CP-VP and CPE-VP, a vertex partition is a CPE-VP must be a CP-VP.*

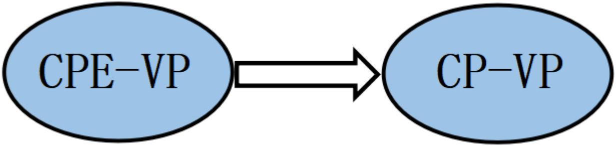



Figure 5: the relationship between CP-VP and CPE-VP

# Graphic condition for the ODD of BNs

## Theorem 2.4 [1]

The ODD of the BN

$$\begin{aligned} x(t+1) &= F\xi(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{21}$$

is implemented if and only if the vertex-colored state transition graph of the BN has a CP-VP $\mathcal{S} = \{S_l\}_{l=1}^{\mu}$, where $\mu$ is some positive integer.

## Remark

*Compared with the algebraic conditions, the partition condition is simply and intuitively.*

[1] Y. Li, J. Zhu, B. Li, Y. Liu and J. Lu, A Necessary and Sufficient Graphic Condition for the Original Disturbance Decoupling of Boolean Networks, IEEE Transactions on Automatic Control, vol. 66, no. 8, pp. 3765-3772, 2021.

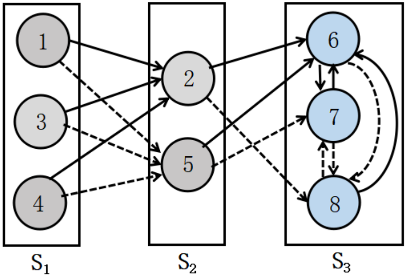



Figure 6: the vertex-colored state transition graph

# Graphic condition for ODD

Proof. (Sufficiency) Denote by $\mathcal{P}=\{P_i\}_{i=1}^{s}$ a CP-VP of the vertex-colored state transition graph of the BN. Since $\mathcal{P}$ is perfect, for any $l_0 = 1, 2, \dots, s$, there exist $l_1, l_2, \dots, l_t, \dots$ such that

$$\mathcal{N}(P_{l_0}) \subset P_{l_1}, \ \mathcal{N}(P_{l_1}) \subset P_{l_2}, \dots, \mathcal{N}(P_{l_{t-1}}) \subset P_{l_t}, \dots. \tag{22}$$

Since $\mathcal{P}$ is concolorous, all the vertices in each $P_{l_k}$ correspond to the same output, i.e., for any given $P_{l_k}$ there exists $c_{l_k}$ such that

$$H\delta_{2^n}^{j} = \delta_{2^r}^{c_{l_k}}, \quad \forall\, j \in P_{l_k}. \tag{23}$$

Let the initial state $x(0) = \delta_{2^n}^{k_0}$, where $k_0 \in P_{l_0}$. For any given disturbance sequence $w(0), w(1), w(2), \dots,$ denote the system state at time $t$ by $x(t) = \delta_{2^n}^{k_t}$, $t \geq 0$. By (22), we have $k_t \in P_{l_t}$. So $H\delta_{2^n}^{k_t} = \delta_{2^r}^{c_{l_t}}$ due to (23). Thus we conclude that the output sequence $\{\delta_{2^r}^{c_{l_i}}\}_{i=1}^{+\infty}$ is independent of the disturbance sequence.

(Necessity). It follows from the algebraic form of the BN that

$$y(s)=HLw(s-1)Lw(s-2)\cdots Lw(1)Lw(0)x(0). \tag{24}$$

Since the original disturbance decoupling is implemented, $y(s)$ is the same for any disturbances $w(0)$, $w(1)$,...,$w(s-1)$. Thus, letting $w(t)=\delta_{2^m}^{i_t}$ for any $t\geq 0$, we can see that

$$HL_{i_{s-1}}L_{i_{s-2}}\cdots L_{i_1}L_{i_0}x(0) \tag{25}$$

is the same for any $1\leq i_1,i_2,\ldots,i_{s-1}\leq 2^m$. By (25) and the arbitrariness of $x(0)$, we have

$$HL_{i_{s-1}}L_{i_{s-2}}\cdots L_{i_1}L_{i_0}=HL_1^s \tag{26}$$

for any $1\leq i_1,i_2,\ldots,i_{s-1}\leq 2^m$ and $s\geq 0$. We write

$$\mathcal{O}_\mu := \begin{bmatrix} H \\ HL_1 \\ HL_1^2 \\ \vdots \\ HL_1^{\mu-1} \end{bmatrix} = \begin{bmatrix} \delta_{2^r}[h_{11}\ h_{12}\ \ldots\ h_{12^n}] \\ \delta_{2^r}[h_{21}\ h_{22}\ \ldots\ h_{22^n}] \\ \delta_{2^r}[h_{31}\ h_{32}\ \ldots\ h_{32^n}] \\ \vdots \\ \delta_{2^r}[h_{\mu1}\ h_{\mu2}\ \ldots\ h_{\mu2^n}] \end{bmatrix}. \tag{27}$$

There exists $\mu^*$ such that

$$HL_1^\mu\in\{H,\ HL_1,\ldots,HL_1^{\mu^*-1}\},\ \forall\ \mu\geq\mu^*. \tag{28}$$

The matrix $\mathcal{O}_{\mu^*}$ is just the observability matrix.

Construct a vertex partition $\mathcal{P} = \{P_l\}_{l=1}^{\eta}$ of $V$ following such a way that $a$ and $b$ belong to the same class of partition $\mathcal{P}$ if and only if $\text{Col}_a(\mathcal{O}_{\mu^*}) = \text{Col}_b(\mathcal{O}_{\mu^*})$. From the construction of $\mathcal{P} = \{P_l\}_{l=1}^{\eta}$, for any $a, b \in P_l$, we have $\text{Col}_a(H) = \text{Col}_b(H)$, which implies that $H\delta_{2^n}^a = H\delta_{2^n}^b$, i.e., $a$ and $b$ have the same color. So the vertex partition $\mathcal{P} = \{P_l\}_{l=1}^{\eta}$ is concolorous.

We claim that the vertex partition $\mathcal{P} = \{P_l\}_{l=1}^{\eta}$ is also perfect, i.e., for any class $P_l$, its out-neighborhood $\mathcal{N}(P_l)$ is a subset of some class $P_{\alpha_l}$. To prove it, for arbitrary $a, b \in P_l$, we consider all the out-neighbors of $a$ and $b$. From the first equation of (21), it follows that the out-neighborhoods can be written as

$$\mathcal{N}(a) = \{a_p \mid \delta_{2^n}^{a_p} = L_p\delta_{2^n}^{a}, p = 1, 2, \cdots 2^m.\}, \tag{29}$$

$$\mathcal{N}(b) = \{b_q \mid \delta_{2^n}^{b_q} = L_q\delta_{2^n}^{b}, q = 1, 2, \cdots 2^m.\}. \tag{30}$$

For any $a, b \in P_l$, it follows from the construction of $\mathcal{P}$ that

$$HL_1^t \delta_{2^n}^a = HL_1^t \delta_{2^n}^b, \quad \forall\ t = 1, 2, \ldots. \tag{31}$$

Applying (26) to (31), we have

$$HL_1^{t-1} L_p \delta_{2^n}^a = HL_1^{t-1} L_q \delta_{2^n}^b, \quad \forall\ t = 1, 2, \ldots. \tag{32}$$

Using (29) and (30) to (32) yields

$$HL_1^{t-1} \delta_{2^n}^{a_p} = HL_1^{t-1} \delta_{2^n}^{b_q}, \quad \forall\ t = 1, 2, \ldots, \tag{33}$$

whichimplies that $a_p$ and $b_q$ are in the same class of $\mathcal{P}$. By the arbitrariness of $a$ and $b$ in $\mathcal{P}_l$, we conclude that there exists $\alpha_l$ such that $\mathcal{P}_l \subset \mathcal{P}_{\alpha_l}$.

# Partition condition for the DD of BNs

## Theorem 2.5 [1]

There exists a logical coordinate transformation $z = Tx$ such that the BN

$$\begin{aligned} x(t+1) &= F\xi(t)x(t), \\ y(t) &= Hx(t), \end{aligned} \tag{34}$$

becomes

$$\begin{aligned} z^{[1]}(t+1) &= G_1\xi(t)z(t), \\ z^{[2]}(t+1) &= G_2 z^{[2]}(t), \\ y(t) &= Ez^{[2]}(t), \end{aligned} \tag{35}$$

if and only if the vertex-colored state transition graph $\mathcal{G}$ of BN has a CPE-VP $\mathcal{S} = \{S_i\}_{i=1}^{2^{n-s}}$ with $|S_i| = 2^s$.

[1] Y. Li, J. Zhu, B. Li, Y. Liu and J. Lu, A Necessary and Sufficient Graphic Condition for the Original Disturbance Decoupling of Boolean Networks, IEEE Transactions on Automatic Control, vol. 66, no. 8, pp. 3765-3772, 2021.

# The relationship between DD and ODD

## Remark

*From the above two Theorems, if one system is DD, the system must be ODD. However, the converse is not true.*

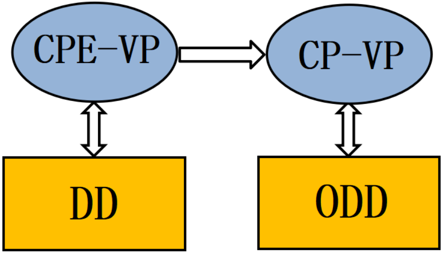



Figure 7: the relationship between DD and ODD

# Algorithm

- We provided an algorithm for computing CP-VP in [1]. (1). compute the finest P-VP; (2). check it to see if it's a CP-VP.
- For the disturbance decoupling dependent on decomposition, one needs to search for an E-VP $\mathcal{P}$ satisfying $\mathcal{K} \sqsubset \mathcal{P} \sqsubset \mathcal{C}$, where $\mathcal{K}$ is the finest CP-VP obtained from Algorithm 2 in [1] and $\mathcal{C}$ is the coarsest C-VP corresponding to the assigned colors of the vertices.

[1] Y. Li, J. Zhu, B. Li, Y. Liu and J. Lu, A Necessary and Sufficient Graphic Condition for the Original Disturbance Decoupling of Boolean Networks, IEEE Transactions on Automatic Control, vol. 66, no. 8, pp. 3765-3772, 2021.

# An illustrative example

## Example

Consider the BN with disturbance described by

$$\begin{aligned} x_1(t+1) =& \xi(t) \wedge x_1(t) \wedge (x_2(t) \to x_3(t)), \\ x_2(t+1) =& \xi(t) \wedge [x_1(t) \vee \neg x_1(t) \wedge (x_2(t) \to x_3(t))] \vee \\ & \neg\xi(t) \wedge [x_1(t) \wedge (x_2(t) \to x_3(t))], \\ x_3(t+1) =& \xi(t) \wedge \neg x_1(t) \wedge x_2(t) \wedge \neg x_3(t) \vee \neg\xi(t) \wedge \\ & [x_1(t) \wedge (x_2(t) \to x_3(t)) \vee \neg x_1(t) \wedge (x_2(t) \leftrightarrow x_3(t))], \\ y(t) =& x_1(t) \vee \neg x_1(t) \wedge x_2(t) \wedge x_3(t), \end{aligned} \tag{36}$$

where $x_1, x_2, x_3, \xi, y \in \Delta_2$. Let $x = x_1 x_2 x_3$. Then the algebraic form of the system (36) is

$$\begin{aligned} x(t+1) &= F\xi(t)x(t), \\ y(t) &= Hx(t) \end{aligned}$$

with $F = \delta_8 = [2, 6, 2, 2, 6, 7, 6, 6, 5, 8, 5, 5, 7, 8, 8, 7]$ and $H = \delta_2[1, 1, 1, 1, 1, 2, 2, 2]$.

# An illustrative example

The vertex-colored STG is shown in the following, we can easily see that the ODD is implemented and the vertex-colored state transition graph has a CP-VP $\mathcal{S} = \{\ S_1 = \{1,3,4\},\ \ S_2 = \{2,5\},\ \ S_3 = \{6,7,8\}\}$.

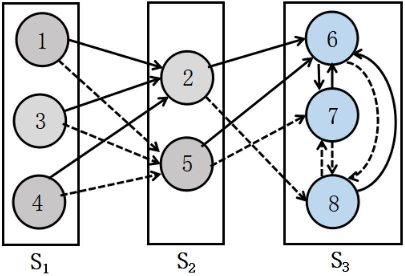



Figure 8: the vertex-colored state transition graph

- How to design controller?

- How to solve DDP with low computational complexity?

# Outline

# System model

Consider the BCN described as follows:

$$\begin{cases} X_1(t+1) = f_1(X_1(t), \ldots, X_n(t), U_1(t), \ldots, U_m(t)), \\ \vdots \\ X_n(t+1) = f_n(X_1(t), \ldots, X_n(t), U_1(t), \ldots, U_m(t)), \\ Y_i(t) = h_i(X_1(t), \ldots, X_n(t)), \ i = 1, 2 \cdots, m, \end{cases} \quad (37)$$

where the input and the output are assumed to have the same number of channels for convenience, the state vector is $X = [X_1, X_2, \ldots, X_n]^{\mathrm{T}} \in \mathcal{B}^n$, the control vector is $U = [U_1, U_2, \ldots, U_m]^{\mathrm{T}} \in \mathcal{B}^m$ and the output vector is $Y = [Y_1, Y_2, \ldots, Y_m]^{\mathrm{T}} \in \mathcal{B}^m$. BCN (37) can be expressed in the algebraic form (Cheng & Qi, 2010) as follows:

$$\begin{cases} \mathbf{x}(t+1) = L\mathbf{u}(t)\mathbf{x}(t), \\ \mathbf{y}_i(t) = H_i\mathbf{x}(t), i = 1, 2, \cdots, m, \end{cases} \quad (38)$$

where $L = [L_1, L_2, \ldots, L_{2^m}] \in \mathcal{L}_{2^n \times 2^{m+n}}$, $L_k = L\delta_{2^m}^k \in \mathcal{L}_{2^n \times 2^n}$, $H_i \in \mathcal{L}_{2 \times 2^n}$, $\mathbf{x} = \ltimes_{i=1}^n \mathbf{x}_i = \mathbb{C}_n(X)$, $\mathbf{u} = \ltimes_{i=1}^m \mathbf{u}_i = \mathbb{C}_m(U)$, and $\mathbf{y}_i = \mathbb{C}_2(Y_i)$.

## Definition 3.1 [1-3] IOD dependent on decomposition

The BCN (38) with inputs and outputs having the same cardinality, $m$, is said to be input-output decoupled if there exists a logical coordinate transformation $\mathbf{z} = T\mathbf{x}$ such that, under $\mathbf{z}$ coordinate, (38) can be decomposed into the following form

$$\begin{cases} \mathbf{z}^1(t+1) = F_1\mathbf{u}_1(t)\mathbf{z}^1(t), \\ \vdots \\ \mathbf{z}^m(t+1) = F_m\mathbf{u}_m(t)\mathbf{z}^m(t), \\ \mathbf{z}^{m+1}(t+1) = F_{m+1}\mathbf{u}(t)\mathbf{z}(t), \\ \mathbf{y}_1(t) = E_1\mathbf{z}^1(t), \\ \vdots \\ \mathbf{y}_m(t) = E_m\mathbf{z}^m(t), \end{cases} \tag{39}$$

where $\mathbf{u} = \mathbf{u}_1\mathbf{u}_2\cdots\mathbf{u}_m \in \mathcal{L}_{2^m}$, $\mathbf{z} = \mathbf{z}^1\mathbf{z}^2\cdots\mathbf{z}^{m+1} \in \mathcal{L}_{2^n}$, $\mathbf{z}^l \in \mathcal{L}_{2^{n_l}}$, $F_l \in \mathcal{L}_{2^{n_l}\times 2^{n_l+1}}$, $l = 1, 2, \cdots, m+1$, $n = n_1 + n_2 + \cdots + n_{m+1}$, and $E_i \in \mathcal{L}_{2\times 2^{n_i}}$, $i = 1, 2, \cdots, m$.

[1] S. Fu, J. Zhao, and J. Wang, Input-output decoupling control design for switched Boolean control networks, Journal of the Franklin Institute, vol. 355, no. 17, pp. 8576-8596, 2018.
[2] J. Pan, J. Feng, J. Yao, and J. Zhao, Input-output decoupling of Boolean control networks, Asian Journal of Control, vol. 20, no. 6, pp. 2185-2194, 2018.
[3] D. Cheng, H. Qi and Z. Li, Analysis and control of Boolean networks: a semi-tensor product

## Definition 3.2 [1] IOD free of decomposition

The BCN (37) with inputs and outputs having the same cardinality, $m$, is said to be input-output decoupled if for every index $i \in [1, m]$ and every initial state $X(0) \in \mathcal{B}^n$, if $U(t)$ and $\hat{U}(t)$, $t \in \mathbb{Z}_+$, are two input sequences characterized by the fact that their $i$th entries coincide at every time instant, i.e.

$$U_i(t) = \hat{U}_i(t), \ \forall \ t \in \mathbb{Z}_+,$$

then theoutput sequences $Y(t)$ and $\hat{Y}(t)$, $t \in \mathbb{Z}_+$, generated by BCN (37) corresponding to $X(0)$, $U(t)$ and $\hat{U}(t)$, $t \in \mathbb{Z}_+$, respectively, satisfy

$$Y_i(t) = \hat{Y}_i(t), \ \forall \ t \in \mathbb{Z}_+.$$

[1] M. Valcher, Input/output decoupling of boolean control networks. IET Control Theory & Applications, vol.11, no.13, pp: 2081-2088, 2017.

## Remark 3.1

According to Definition 3.1, if a BCN is input-output decoupled, then the BCN must be decomposed into form (39) under a logical coordinate transformation, but this constraint is removed in Definition 3.2. That is to say, compared with Definition 3.1, Definition 3.2 is more general.

Besides, Definition 3.2 captures the original meaning of input-output decoupling, i.e., each output channel can only be possibly controlled by one input channel, which does not related to any system decomposition. Actually, it is possible to construct a BCN which is input-output decoupled but can not be decomposed into form (39).

# An illustrative example

## Example

consider the BCN

$$\begin{array}{rcl} X_1(t+1) & = & U_1(t) \wedge X_1(t), \\ X_2(t+1) & = & (\neg U_1(t) \wedge X_1(t)) \vee X_2(t) \vee U_2(t), \\ Y_1(t) & = & X_1(t), \\ Y_2(t) & = & X_1(t) \vee X_2(t), \end{array} \tag{40}$$

where $X_1, X_2, U_1, U_2 \in \mathscr{B}$, $(X_1, X_2)^{\mathrm{T}}$ is the state vector, $(U_1, U_2)^{\mathrm{T}}$ is the input vector, $(Y_1, Y_2)^{\mathrm{T}}$ is the output vector. It is easy to check that

$$\begin{array}{rcl} Y_1(t) & = & U_1(t-1) \wedge U_1(t-2) \wedge \ldots \wedge U_1(0) \wedge X_1(0), \\ Y_2(t) & = & U_2(t-1) \vee U_2(t-2) \vee \ldots \vee U_2(0) \vee X_1(0) \vee X_2(0). \end{array}$$

So BCN (40) is input-output decoupled. However, it is impossible to decompose BCN (40) into two independent subsystems to realize input-output decoupling.

# An illustrative example

Indeed, otherwise, there is a logical coordinate transformation $Z = \phi(X)$ such that, in the $Z$ coordinate frame, BCN (40) becomes

$$\begin{array}{rcl} Z_1(t+1) & = & f_1(Z_1(t), U_1(t)), \\ Z_2(t+1) & = & f_2(Z_2(t), U_2(t)), \\ Y_1(t) & = & h_1(Z_1(t)), \\ Y_2(t) & = & h_2(Z_2(t)). \end{array} \tag{41}$$

All the possible expressions of $h_2(Z_2)$ are

$$h_2(Z_2)=1, h_2(Z_2)=0, h_2(Z_2)=Z_2, h_2(Z_2)=\neg Z_2. \tag{42}$$

When $Y_2 = h_2(Z_2) = 1$ or $Y_2 = h_2(Z_2) = 0$, all the possible states yield the same $Y_2$. When $Y_2 = h_2(Z_2) = Z_2$ or $Y_2 = h_2(Z_2) = \neg Z_2$, a half of states must have output $Y_2 = 1$ and the other half must have output $Y_2 = 0$. That is to say, the number of states that have output $Y_2 = 0$ is 0 or 2 or 4. However, in the $X$ coordinate frame, by

$$Y_2 = X_1 \vee X_2 = \begin{cases} 0, & (X_1, X_2) = (0,0), \\ 1, & \text{otherwise}, \end{cases} \tag{43}$$

we conclude that only one state has output $Y_2 = 0$. A contradiction.

# Algebraic and graphic conditions for IOD

Consider BCN (38)

$$\begin{cases} \mathbf{x}(t+1) = L\mathbf{u}(t)\mathbf{x}(t), \\ \mathbf{y}_i(t) = H_i\mathbf{x}(t), i = 1, 2, \cdots, m, \end{cases}$$

If we assign the $i$th input channel as $u_i = \delta_2^b$ $(b = 1, 2)$, then BCN (38) becomes

$$x(t+1) = Lu_1(t)\cdots u_{i-1}(t)\delta_2^b u_{i+1}(t)\cdots u_m(t)x(t). \tag{44}$$

Similarly, we get an adjacency matrix of the STG of (44) as follows:

$$M_{ib} = \frac{1}{2^{m-1}} L\mathbf{1}_{2^{i-1}}\delta_2^b\mathbf{1}_{2^{m-i}}. \tag{45}$$

Denote the vertex-colored STG associated with $H_i$ and $M_{ib}$ by $\mathcal{G}_{ib}$, $b = 1, 2$.

# Algebraic and graphic conditions for IOD

## Theorem 3.1 [1]

BCN (38) is input-output decoupled according to Definition 2.2 if and only if for every $i = 1, 2, \ldots, m$, every $t \geq 1$ and every $b_1, b_2, \ldots, b_t \in \{1, 2\}$, the matrix

$$H_i M_{ib_t} \cdots M_{ib_1} \in \mathcal{L}_{2\times 2^n},$$

where $M_{ib_s}$, $s = 1, 2, \cdots, t$, defined in (45).

## Theorem 3.2 [1]

BCN (38) is input-output decoupled according to Definition 2.2 if and only if for every $i \in [1, m]$, $\mathcal{G}_{i1}$ and $\mathcal{G}_{i2}$ have a common CP-VP.

[1] Y. Li and J. Zhu, Necessary and sufficient vertex partition conditions for input-output decoupling of Boolean control networks, Automatica, vol. 137, p. 110097, 2022.

# Algebraic and graphic conditions for IOD

## Theorem 3.3 [1]

BCN (38) is input-output decoupled according to Definition 2.1 if and only if
(I) for every $i \in [1, m]$, $\mathcal{G}_{i1}$ and $\mathcal{G}_{i2}$ have a common CPE-VP denoted by $\mathcal{S}^i = \{S^i_l\}_{l=1}^{2^{n_i}}$;
(II) $\mathcal{S}^{1\cdots m} := \mathcal{S}^1 \bar{\wedge} \mathcal{S}^2 \bar{\wedge} \cdots \bar{\wedge} \mathcal{S}^m$ is an equal vertex partition with $|\mathcal{S}^{1\cdots m}| = 2^{N_m}$.

## Remark 3.2

By using the vertex partition conditions, Theorems 3.2 and 3.3 clearly reveal the essential difference between Definition 3.1 and Definition 3.2. Definition 3.2 is equivalent to the existence of some CP-VPs, but Definition 3.1 further requires that the CP-VPs and their greatest common refinement are equal partitions.

[1] Y. Li and J. Zhu, Necessary and sufficient vertex partition conditions for input-output decoupling of Boolean control networks, Automatica, vol. 137, p. 110097, 2022.

# An illustrative example

## Example

Consider the following BCN, which is a reduced sub-network of signal transduction networks (Li, Assmann, & Albert, 2006):

$$\begin{aligned} X_1(t+1) =& U_1(t) \wedge \neg X_3(t) \wedge U_2(t), \\ X_2(t+1) =& U_2(t), \\ X_3(t+1) =& \neg(X_2(t) \vee U_1(t)) \wedge U_2(t), \end{aligned} \tag{46}$$

where $X_1$, $X_2$, $X_3 \in \mathcal{B}$ are state variables denoting *Atrboh, Ros and ABL1*, respectively, $U_1$, $U_2 \in \mathcal{B}$ are *the external inputs*. The output equations are given by

$$\begin{aligned} Y_1(t) =& [X_1(t) \wedge (X_2(t) \to X_3(t))] \vee (\neg X_1(t) \wedge \neg X_2(t) \wedge X_3(t)), \\ Y_1(t) =& [X_1(t) \wedge (X_2(t) \to X_3(t))] \vee (\neg X_1(t) \wedge \neg X_2(t)), \end{aligned} \tag{47}$$

where $Y_1, Y_2 \in \mathcal{B}$.

# An illustrative example

Logical equations (59) and (68) can be converted into the algebraic form (38) with $\mathbf{x} \in \mathscr{L}_8$, $\mathbf{u} \in \mathscr{L}_4$, $\mathbf{y} \in \mathscr{L}_2$,

$$
\begin{aligned}
L =& \delta_8[6\ 2\ 6\ 2\ 6\ 2\ 6\ 2\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8 \\
& \quad 6\ 6\ 5\ 5\ 6\ 6\ 5\ 5\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8], \\
H_1 =& \delta_2[1\ 2\ 1\ 1\ 2\ 2\ 1\ 2], \\
H_2 =& \delta_2[1\ 2\ 1\ 1\ 2\ 2\ 1\ 1].
\end{aligned}
$$

Let $L = [L_1, L_2, L_3, L_4]$ with each $L_i \in \mathscr{L}_{8\times 8}$. From $L$, $H_1$, $H_2$, we have

$$M_{11} = \frac{1}{2}(L_1 + L_2), M_{12} = \frac{1}{2}(L_3 + L_4),$$

$$M_{21} = \frac{1}{2}(L_1 + L_3), M_{22} = \frac{1}{2}(L_2 + L_4).$$

Let $\mathbf{y}_1 = \delta_2^1$ $(\delta_2^2)$ represent gray (white), $\mathbf{y}_2 = \delta_2^1$ $(\delta_2^2)$ represent green (white).

# An illustrative example

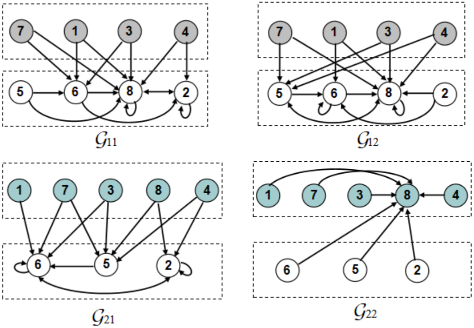



Figure 9: the vertex-colored STG $\mathcal{G}_{11}$, $\mathcal{G}_{12}$, $\mathcal{G}_{21}$, $\mathcal{G}_{22}$.

$\mathcal{G}_{11}$ and $\mathcal{G}_{12}$ have a common CP-VP as $\mathcal{P}^1 = \{\{1,3,4,7\},\{2,5,6,8\}\}$;
$\mathcal{G}_{21}$ and $\mathcal{G}_{22}$ have a common CP-VP as $\mathcal{P}^2 = \{\{1,3,4,7,8\},\{2,5,6\}\}$.
According to Theorem 2.2, the input-output decoupling defined by Definition 2.2 of BCN (59) is realized. However, $\mathcal{G}_{21}$ and $\mathcal{G}_{22}$ have no common CPE-VP. Hence, by Theorem 2.3, the BCN is not input-output decoupling according to Definition 2.1.

- Input-output decoupling;
- Block decoupling [1,2];
- Morgan problem [3];

[1] Y. Yu, J. Feng, J. Pan, and D. Cheng, Block decoupling of Boolean control networks, IEEE Transactions on Automatic Control, 64(8), pp. 3129-3140, 2018.

[2] L. Wang, Y. Li, and J. Zhu, On block-decoupling of Boolean control networks, International Journal of Control, Automation, and Systems 21(1), pp.40-51, 2023.

[3] S. Fu, Y. Wang, and D. Cheng, et al. Morgan's problem of Boolean control networks, Control Theory Technol. 15, 316-326 (2017).

# Some work on the IOD of BCNs

- S. Fu, J. Zhao, and J. Wang, Input-output decoupling control design for switched Boolean control networks, Journal of the Franklin Institute, vol. 355, no. 17, pp. 8576-8596, 2018.
- J. Pan, J. Feng, J. Yao, and J. Zhao, Input-output decoupling of Boolean control networks, Asian Journal of Control, vol. 20, no. 6, pp. 2185-2194, 2018.
- M. Valcher, Input/output decoupling of Boolean control networks. IET Control Theory & Applications, vol.11, no.13, pp: 2081-2088, 2017.
- Y. Li and J. Zhu, Necessary and sufficient vertex partition conditions for input-output decoupling of Boolean control networks, Automatica, vol. 137, p. 110097, 2022.

# Outline

# System decomposition w.r.t. inputs

## Definition 4.1 [1]

Consider the BCN

$$x(t+1) = Lu(t)x(t), \tag{48}$$

where $L \in \mathcal{L}_{2^n \times 2^{m+n}}$. The BCN is said to be decomposable with respect to inputs with order $n-s$, if there exists a logical coordinate transformation $z_i = g_i(x_1, ..., x_n)(i = 1, 2, ..., n)$ such that (48) becomes

$$\begin{aligned} z^1(t) &= G_1 u(t) z(t)), \\ z^2(t) &= G_2 z^2(t), \end{aligned} \tag{49}$$

where $z^1 = z_1 z_2 \cdots z_s$, $z^2 = z_{s+1} z_{s+2} \cdots z_n$.

[1] Y. Zou, J. Zhu, System decomposition with respect to inputs for Boolean control networks, *Automatica*, 50(4),1304-1309, 2014.

## Theorem 4.1 [1]

Consider the BCN

$$x(t+1) = Lu(t)x(t). \tag{50}$$

The BCN is decomposable with respect to inputs with order $n-s$ if and only if the state transition graph has a PEVP $\{P_i\}_{i=1}^{2^{n-s}}$ with each $|P_i| = 2^s$

## Remark 4.1

- A Vertex Set Uniting Algorithm was proposed in [1] for finding a PEVP.
- A decomposition with respect to inputs of maximum order is called the maximum decomposition with respect to inputs.
- Assume that the largest uncontrollable subspace is a regular subspace.

$$\begin{aligned} z^1(t) &= G_1 u(t) z(t)), \\ z^2(t) &= G_2 z^2(t), \end{aligned} \tag{51}$$

is called the normal controllable form [2].

[1] Y. Zou, J. Zhu, System decomposition with respect to inputs for Boolean control networks, Automatica, 50(4),1304-1309, 2014.
[2] D. Cheng, Z. Li, and H. Qi, Realization of Boolean control networks, Automatica, vol. 46, no. 1, pp. 62-69, 2010.

# Observability Decomposition

The observability decomposition is one of the fundamental issues in the control system theory, which divides a system into the observable and the unobservable subsystems [1,2].

[1] R. Kalman, Mathematical description of linear dynamical systems, *SIAM J. Appl. Math.*, vol. 1, no. 2, pp.152-192, 1963.
[2] Y. Kawano and Ü. Kotta, Single-experiment observability decomposition of discrete-time analytic systems, *Syst. Control Lett.*, vol. 97, pp. 193-199, 2016.

# System Model

Consider the BCN

$$\begin{array}{rcl} X(t+1) & = & f(X(t), U(t)), \\ Y(t) & = & h(X(t)), \end{array} \tag{52}$$

where $X = [X_1, X_2, \ldots, X_n]^{\mathrm{T}} \in \mathscr{B}^n$, $U = [U_1, U_2, \ldots, U_m]^{\mathrm{T}} \in \mathscr{B}^m$ , $Y = [Y_1, Y_2, \ldots, Y_p]^{\mathrm{T}} \in \mathscr{B}^p$.

Algebraic form of (52)[1].

$$\begin{array}{rcl} x(t+1) & = & Lu(t)x(t), \\ y(t) & = & Hx(t), \end{array} \tag{53}$$

where $x \in \Delta_{2^n}$, $y \in \Delta_{2^p}$, $L \in \mathcal{L}_{2^n \times 2^{m+n+q}}$ and $H \in \mathcal{L}_{2^p \times 2^n}$.

[1] D. Cheng, H. Qi, and Z. Li, *Analysis and control of Boolean control networks: a semi-tensor product approach*, Springer, London, 2011.

# Background

- **the normal observable form** [1]
  - regularity of the largest unobservable subspace
  - there exists an expression of (2) which have largest unobservable subspace $z^2$ as

$$\begin{array}{rcl} z^{[1]}(t+1) & = & G_1 u(t) z^{[1]}(t), \\ z^{[2]}(t+1) & = & G_2 u(t) z(t), \\ y(t) & = & M z^{[1]}(t). \end{array} \tag{54}$$

- **the maximum decomposition w.r.t. outputs** [2]
  - without regularity hypothesis
  - does not involve the observability of the remaining part


[1] D. Cheng, Z. Li, and H. Qi, Realization of Boolean control networks, *Automatica*, vol. 46, no. 1, pp. 62-69, 2010.
[2] Y. Zou and J. Zhu, Graph theory methods for decomposition w.r.t. outputs of Boolean control networks, *J. Syst. Sci. and Complex.*, vol. 30, no. 3, pp. 519-534, 2017.

# Problem Statement

**Definition 4.2 [1] (Observability)**

Consider a BCN. Two states $x$ and $\bar{x}$, are said to be indistinguishable, if for any input sequence $\{u(0), u(1), \cdots,\}$, the two output sequences $\{y(0), y(1), \cdots\}$ and $\{\bar{y}(0), \bar{y}(1), \cdots\}$, corresponding to the initial states $x(0) = x$ and $x(0) = \bar{x}$ respectively are same; otherwise they are distinguishable.

The BCN is said to be observable if any two distinct states are distinguishable.

[1] Y. Zhao, H. Qi, and D. Cheng, Input-state incidence matrix of Boolean control networks and its applications, Syst. Control Lett., vol. 59, no. 12, pp. 767-774, 2010.

# Problem Statement

## Definition 4.3 [1] (Observability decomposition)

Consider a BCN. The observability decomposition is said to be realizable if there is a coordinate transformation $z = Tx$ such that under the $z$ coordinate frame the BCN becomes

$$\begin{aligned} z^{[1]}(t+1) &= G_1 u(t) z^{[1]}(t), \\ z^{[2]}(t+1) &= G_2 u(t) z(t), \\ y(t) &= M z^{[1]}(t), \end{aligned} \tag{55}$$

and $z^{[1]}$ subsystem is observable by Definition 2.1, where $z^{[1]} \in \Delta_{2^{n_1}}$, $z^{[2]} \in \Delta_{2^{n_2}}$, $z = z^{[1]} z^{[2]}$.

[1] Y. Li, J. Zhu, Observability decomposition of Boolean control networks. IEEE Trans. Autom. Control, 68(2):1267-1274, 2023.

# Main Results

## Proposition 4.1

Indistinguishable relation on $V = \{1, 2, \cdots, 2^n\}$ is an equivalence relation, which induces an associated partition denoted by $\mathcal{P} = \{P_l\}_{l=1}^{s}$ of $V$, where each $P_l$ is an equivalence class, that is, for any $l$, for any two vertices $p, q \in P_l$, $p, q$ are indistinguishable and for any $l_1 \neq l_2$, for any $p \in P_{l_1}, q \in P_{l_2}$, $p, q$ are distinguishable.

## Proposition 4.2

The partition $\mathcal{P} = \{P_l\}_{l=1}^{s}$ induced by the indistinguishable relation on $V = \{1, 2, \cdots, 2^n\}$ is the coarsest common CP-VP of $\mathcal{G}_1, \cdots, \mathcal{G}_{2^m}$.

[1] Y. Li, J. Zhu, Observability decomposition of Boolean control networks. IEEE Trans. Autom. Control, 68(2):1267-1274, 2023.

# Main Results

## Corollary 4.1

BCN (53) is observable if and only if the coarsest common CP-VP of $\mathcal{G}_1, \cdots, \mathcal{G}_{2^m}$ is the single point partition.

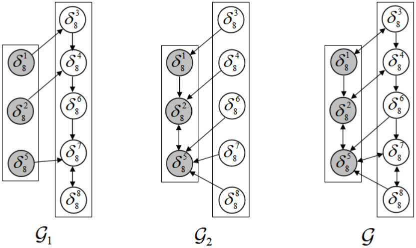



Figure 10: CCP-VP

# Main Results

## Theorem 4.1 [1]

Assume that $n_1 > 0$ and $n_1 + n_2 = n$. Then BCN (53) has an observability decomposition form

$$\begin{array}{rcl} z^{[1]}(t+1) & = & G_1 u(t) z^{[1]}(t), \\ z^{[2]}(t+1) & = & G_2 u(t) z(t), \\ y(t) & = & M z^{[1]}(t), \end{array}$$

where $z^{[1]} \in \Delta_{2^{n_1}}$, $z^{[2]} \in \Delta_{2^{n_2}}$, $z = z^{[1]} z^{[2]}$ if and only if

(I) $\mathcal{G}_1, \cdots, \mathcal{G}_{2^m}$ have a common CPE-VP $\mathcal{P} = \{P_l\}_{l=1}^{2^{n_1}}$ with $|P_l| = 2^{n_2}$;

(II) $\mathcal{P}$ is the coarsest common CP-VP.

## Corollary 4.2 [1]

BCN (53) has an observability decomposition if and only if the coarsest common CP-VP of $\mathcal{G}_1, \cdots, \mathcal{G}_{2^m}$ is an E-VP that is not the single point partition.

[1] Y. Li, J. Zhu, Observability decomposition of Boolean control networks. IEEE Trans. Autom. Control, 68(2):1267-1274, 2023.

Define

$$\begin{aligned} \Gamma_0 &= \{H\}, \\ \Gamma_\mu &= \{HL_{j_\mu} \cdots L_{j_1} \mid j_\mu, \cdots, j_1 \in [1, 2^m]\}, \end{aligned} \tag{56}$$

where $\mu = 1, 2, \ldots$. For notational ease, we also use $\Gamma_\mu$ to denote the matrix consisting of its elements and arranging in a column. For instance,

$$\Gamma_0 = H, \ \Gamma_1 = \begin{bmatrix} HL_1 \\ HL_2 \\ \vdots \\ HL_{2^m} \end{bmatrix}, \ \Gamma_2 = \begin{bmatrix} HL_1 L_1 \\ HL_1 L_2 \\ \vdots \\ HL_{2^m} L_{2^m} \end{bmatrix}.$$

Construct a matrix as

$$\mathcal{O} = \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_{2^n - 1} \end{bmatrix}. \tag{57}$$

# An algorithm

## Proposition 4.3 [1]

Construct a vertex partition $\mathcal{P} = \{P_l\}_{l=1}^{s}$ of $V = \{1, 2, \cdots, 2^n\}$ following such a way that

$$\forall \;\; a, b \in P_l \Leftrightarrow \mathrm{Col}_a(\mathcal{O}) = \mathrm{Col}_b(\mathcal{O}). \tag{58}$$

Then, $\mathcal{P}$ is the coarsest common CP-VP of $\mathcal{G}_1, \cdots, \mathcal{G}_{2^m}$.

[1] Y. Li, J. Zhu, Observability decomposition of Boolean control networks. IEEE Trans. Autom. Control, 68(2):1267-1274, 2023.

# An algorithm

Based on Proposition 4.3, we give the following algorithm.

## Algorithm 4.1 [1]

Consider BCN (52).

**Step 1**. Compute the algebraic form (53).

**Step 2**. Compute (57) and construct the coarsest common CP-VP $\mathcal{P} = \{P_l\}_{l=1}^{s}$ of $\mathcal{G}_1, \cdots, \mathcal{G}_{2^m}$ as (58).

**Step 3**.

- If $\mathcal{P}$ is a single point set partition, by Corollary 4.1, BCN (53) is observable; otherwise BCN (53) is unobservable.
- If $\mathcal{P}$ is an E-VP that is not a single point set partition, by Corollary 4.2, the observability decomposition of BCN (53) is realizable; otherwise it is not.

[1] Y. Li, J. Zhu, Observability decomposition of Boolean control networks. IEEE Trans. Autom. Control, 68(2):1267-1274, 2023.

# An illustrative example

## Example

Consider a simple *cell apoptosis* Boolean model expressed as follows:

$$\begin{aligned} X_1(t+1) =& \neg X_2(t) \wedge U(t), \\ X_2(t+1) =& \neg X_1(t) \wedge X_2(t), \\ X_3(t+1) =& X_2(t) \vee U(t), \end{aligned} \tag{59}$$

where $X_1$, $X_2$, $X_3$, $U \in \mathscr{B}$ represent *inhibitor of apoptosis proteins (IAP) , active caspase 3 (C3a), active caspase 8 (C8a) and Fas ligand* respectively.

The output equation is given by

$$Y(t) = [X_1(t) \wedge (X_2(t) \vee X_3(t))] \vee (\neg X_1(t) \wedge \neg X_2(t) \wedge X_3(t)). \tag{60}$$

By using STP of matrices, BCN (59) and (68) can be converted into the algebraic form (53) with $L = [L_1 \ L_2]$,

$$\begin{aligned} L =& \delta_8[7\ 7\ 3\ 3\ 5\ 5\ 3\ 3\ 7\ 7\ 8\ 8\ 5\ 5\ 8\ 8], \\ H =& \delta_2[1\ 1\ 1\ 2\ 2\ 2\ 1\ 2]. \end{aligned}$$

# An illustrative example

Let $\mathbf{y} = \delta_2^1$ ($\delta_2^2$) represent gray (white). Then the vertex-colored STG $\mathcal{G}_1$ and $\mathcal{G}_2$ are shown in Fig. 2.

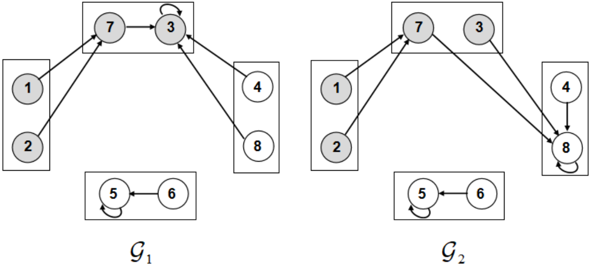



Figure 11: the vertex-colored STG $\mathcal{G}_1$ and $\mathcal{G}_2$.

# An illustrative example

With a straightforward computation, we have

$$\mathcal{O} = \begin{bmatrix} \delta_2[1\ 1\ 1\ 2\ 2\ 2\ 1\ 2] \\ \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 1\ 1] \\ \delta_2[1\ 1\ 2\ 2\ 2\ 2\ 2\ 2] \\ \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 1\ 1] \\ \delta_2[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2] \\ \vdots \\ \delta_2[1\ 1\ 1\ 1\ 2\ 2\ 1\ 1] \\ \delta_2[2\ 2\ 2\ 2\ 2\ 2\ 2\ 2] \end{bmatrix}. \tag{61}$$

By Proposition 4.3, we obtain the coarsest common CP-VP of $\mathcal{G}_1$ and $\mathcal{G}_2$ $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$, where $P_1 = \{1, 2\}, P_2 = \{3, 7\}, P_3 = \{4, 8\}, P_4 = \{5, 6\}$. Since $\mathcal{P}$ is an E-VP that is not a single point set partition, the observability decomposition of the BCN is realizable.

# An illustrative example

By $\mathcal{P}$, one can construct a logical coordinate transformation to realize the observability decomposition. Omitting the computation process, we obtain the observability decomposition form

$$
\begin{aligned}
\mathbf{z}^{[1]}(t+1) =&\delta_4[2\ 2\ 2\ 4\ 2\ 3\ 3\ 4]\mathbf{u}(t)\mathbf{z}^{[1]}(t),\\
\mathbf{z}^{[2]}(t+1) =&\delta_2[2\ 2\ 1\ 1\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 2\ 2\ 2\ 1\ 1]\mathbf{u}(t)\mathbf{z}(t),\\
\mathbf{y}(t) =&\delta_2[1\ 1\ 2\ 2]\mathbf{z}^{[1]}(t),
\end{aligned}
$$

whose logical form is

$$
\begin{array}{rcl}
Z_1(t+1) & = & [U(t) \wedge (Z_1(t) \vee Z_2(t))] \vee (\neg U(t) \wedge \\
 & & Z_1(t) \wedge Z_2(t)),\\
Z_2(t+1) & = & \neg U(t) \wedge (Z_1(t) \bar{\vee} Z_2(t)),\\
Z_3(t+1) & = & [U(t) \wedge (Z_1(t) \wedge \neg Z_2(t) \vee \neg Z_1(t))] \vee \\
 & & (\neg U(t) \wedge \neg Z_1(t) \wedge \neg Z_2(t)),\\
Y(t) & = & Z_1(t).
\end{array}
$$

# Kalman decomposition [1,2]

Consider the BCN

$$\begin{array}{rcl} x(t+1) & = & f(u(t),x(t)), \\ y(t) & = & h(x(t)), \end{array} \qquad (62)$$

where $L \in \mathcal{L}_{2^n \times 2^{m+n}}$. If there exists a logical coordinate transformation such that (62) becomes the form

$$\begin{array}{rcl} Z_1(t+1) & = & F_1(\,Z_1(t), Z_2(t), U(t)\,), \\ Z_2(t+1) & = & F_2(\,Z_2(t)\,), \\ Z_3(t+1) & = & F_3(\,Z_2(t), Z_3(t)\,), \\ Z_4(t+1) & = & F_4(\,Z_1(t), Z_2(t), Z_3(t), Z_4(t), U(t)\,), \\ Y(t) & = & H(\,Z_1(t), Z_2(t)\,), \end{array} \qquad (63)$$

## Definition 5.1 [2]

The system (63) is called the Kalman decomposition of (62) if $n_2 + n_3$ is just the maximum order of the decomposition w.r.t. inputs and $n_3 + n_4$ equals the maximum order of the decomposition w.r.t. outputs.

[1] D. Cheng, Z. Li, and H. Qi, Realization of Boolean control networks, *Automatica*, vol. 46, no. 1, pp. 62-69, 2010.

[2] Y. Zou, J. Zhu, Kalman decomposition for Boolean control networks, *Automatica*, 54, 65-71, 2015.

# Realization

Consider the BCN

$$\begin{array}{rcl} x(t+1) & = & Lu(t)x(t), \\ y(t) & = & Hx(t), \end{array} \tag{64}$$

where $L \in \mathcal{L}_{2^n \times 2^{m+n}}$.

## Definition 6.1 [1]

A logical control system $\Sigma$ :

$$\begin{cases} \tilde{x}(t+1) & = \tilde{L}u(t)\tilde{x}(t), \\ \tilde{y}(t) & = \tilde{H}\tilde{x}(t), \end{cases}$$

is a realization of the BCN (64) if, for any initial state $x_0$ of BCN (64), there is an initial state $\tilde{x}_0$ of $\Sigma$, such that the outputs $\{y(t)\}$ and $\{\tilde{y}(t)\}$ satisfy

$$\tilde{y}(t) = \varphi(y(t)), t = 0, 1, 2, \cdots$$

for any input sequence $u(t)$, $t = 0, 1, 2, \cdots$, where $\varphi$ is a one-to-one correspondence from $\mathrm{Col}(H)$ to $\mathrm{Col}(\tilde{H})$. It is called the minimum realization if the dimension of the state vector of $\Sigma$ is the smallest.

[1] Y. Li, J. Zhu, X. Liu, Results on the realization of Boolean control networks by the vertex partition method, *Science China Information Sciences*, 66(7): 172205, 2023.

# Realization

## Definition 6.2 [1]

Consider the BCN (64). If there exists a subspace $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \cdots, Z_s\}$, such that

$$\begin{cases} z^1(t+1) & = G_1 u(t) z^1(t), \\ y(t) & = K z^1(t), \end{cases} \tag{65}$$

then (65) is a realization of (64), where $Z_i \sim z_i$, $i \in [1, s]$.

In [1], the subspace $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \cdots, Z_s\}$ is a controlled invariant subspace containing $\mathcal{Y}$. Here we call it a $\mathcal{Y}$-friendly controlled invariant subspace, and we call (65) a realization induced by a $\mathcal{Y}$-friendly controlled invariant subspace.

(65) is called the minimum realization if $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \cdots, Z_s\}$ is the smallest $\mathcal{Y}$-friendly controlled invariant subspace. Essentially, $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \cdots, Z_s\}$ in Definition 4.2 is not required to be a regular subspace.

[1] D. Cheng, L. Zhang, and D. Bi, Invariant subspace approach to Boolean (control) networks, IEEE Transactions on Automatic Control, 68(4):2325-2337, 2023.

# Realization

## Definition 6.3 [1]

Consider the BCN (64). Assume that $Z^1 = \mathcal{F}_l(Z_1, Z_2, \cdots, Z_s)$ is a regular subspace and $Z = (Z_1, Z_2, \cdots, Z_n)$ is a new coordinate frame. $Z^1$ is called a $\mathcal{Y}$-friendly controlled invariant regular subspace, if under $Z$, (26) can be expressed as

$$\begin{cases} z^1(t+1) &= G_1 u(t) z^1(t), \\ z^2(t+1) &= G_2 u(t) z(t), \\ y(t) &= K z^1(t). \end{cases} \tag{66}$$

where $K \in \mathcal{L}_{2^p \times 2^s}$, $G_1 \in \mathcal{L}_{2^s \times 2^{s+m}}$, $G_2 \in \mathcal{L}_{2^{n-s} \times 2^{n+m}}$, $z = z^1 z^2$, $z^1 = \ltimes_{i=1}^{s} z_i$, $Z_i \sim z_i$, $i \in [1, n]$.

The subspace $\mathcal{Z}^1 = \mathcal{F}_l\{Z_1, Z_2, \cdots, Z_s\}$ is called a $\mathcal{Y}$-friendly controlled invariant regular subspace, and we call (65) a realization induced by a $\mathcal{Y}$-friendly controlled invariant regular subspace.

[1] D. Cheng, Z. Li, and H. Qi, Realization of Boolean control networks, *Automatica*, vol. 46, no. 1, pp. 62-69, 2010.

# Construct the realization systems

Consider the BCN (64)

$$\begin{array}{rcl} x(t+1) & = & [L_1\ L_2\ \cdots\ L_{2^m}]u(t)x(t), \\ y(t) & = & Hx(t). \end{array} \tag{67}$$

Suppose that $\mathcal{S} = \{S_l\}_{l=1}^r$ is a common CP-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{2^m}$. Using the vertex partition $\mathcal{S}$, we define an equivalence relationship $\sim$ on $\Delta_{2^n}$ as follows: $\delta_{2^n}^i$ and $\delta_{2^n}^j$ are equivalent, i.e., $\delta_{2^n}^i \sim \delta_{2^n}^j$, if and only if $\delta_{2^n}^i$ and $\delta_{2^n}^j$ belong to the same $S_l$, $l = 1, 2, \cdots, r$. The equivalence class of $\delta_{2^n}^i$ denoted by $\tilde{\delta}_{2^n}^i$, is defined by

$$\tilde{\delta}_{2^n}^i := \{\delta_{2^n}^j | \delta_{2^n}^j \sim \delta_{2^n}^i\}.$$

Evidently, for any $\delta_{2^n}^i \in S_l$, $\tilde{\delta}_{2^n}^i = S_l$. Thus, $\Delta_{2^n}/\sim = \mathcal{S} = \{S_l\}_{l=1}^r$ and $|\Delta_{2^n}/\sim| = r$. Because $|\Delta_{2^n}/\sim| = |\Delta_r|$, a one-to-one correspondence $\phi : \Delta_{2^n}/\sim \mapsto \Delta_r$ can be defined as

$$\phi(S_l) = \delta_r^l,\ l = 1, 2, \cdots, r.$$

# Construct the realization systems

With a mild abuse of notation, we still use the symbol $\sim$ to denote the mapping from $\Delta_{2^n}$ to $\Delta_{2^n}/\sim$ induced by the equivalence relationship $\sim$, which is defined as follows: for any $\delta_{2^n}^i \in S_l$, $\sim(\delta_{2^n}^i) = S_l$. Then, we call the composite mapping $\phi\circ\sim: \Delta_{2^n} \mapsto \Delta_r$ the quotient mapping and define

$$z = \phi\circ\sim(x), \ \forall x \in \Delta_{2^n}. \tag{68}$$

We set $\mathrm{Col}(H) = \{\delta_{2^p}^{c_1}, \delta_{2^p}^{c_2}, \cdots \delta_{2^p}^{c_s}\}$. As $|\mathrm{Col}(H)| = s$, we can reduce the order of column of $H$ from $2^p$ to $s$ by defining a one-to-one correspondence $\varphi$ from $\mathrm{Col}(H)$ to $\Delta_s$ as

$$\varphi(\delta_{2^p}^{c_j}) = \delta_s^j, \ j = 1, 2, \cdots, s. \tag{69}$$

Define

$$\tilde{y} = \varphi(y), \ \forall y \in \mathrm{Col}(H). \tag{70}$$

Because there is a one-to-one correspondence between $y$ and $\tilde{y}$, the outputs $y = \delta_{2^n}^{c_j}$ and $\tilde{y} = \delta_s^j$ are seen as the same.

Suppose that $\mathcal{S} = \{S_l\}_{l=1}^{r}$ is a common CP-VP of the vertex-colored STG $\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_{2^m}$ and $\mathrm{Col}(H) = \{\delta_{2^p}^{c_1}, \delta_{2^p}^{c_2}, \cdots \delta_{2^p}^{c_s}\}$. By resorting to the defined mappings $\varphi$ and $\phi \circ \sim$, the quotient logical system can be constructed as

$$\begin{cases} z(t+1) & = \tilde{L}u(t)z(t), \\ \tilde{y}(t) & = \tilde{H}z(t), \end{cases} \tag{71}$$

where $z \in \Delta_r, u \in \Delta_{2^m}, \tilde{y} \in \Delta_s, \tilde{L} \in \mathcal{L}_{r \times r2^m}$ and $\tilde{H} \in \mathcal{L}_{s \times r}$.The defined mappings and the relationship between the quotient logical system and the original system are shown in Fig. 11.

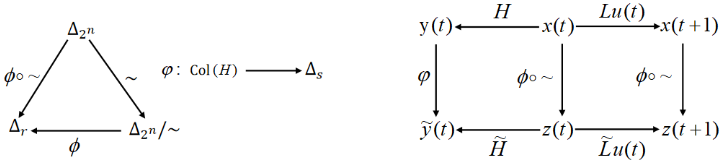


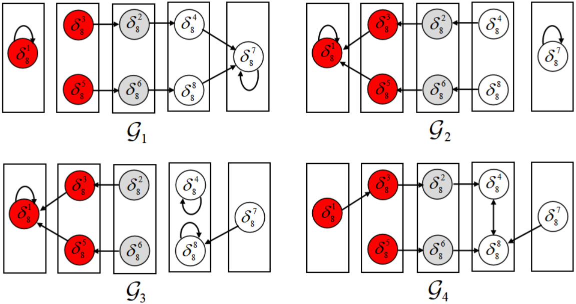

Figure 12:

## Example

Consider a BCN with an algebraic form

$$x(t+1) = Lu(t)x(t), \qquad y(t) = Hx(t), \tag{72}$$

where
$L = \delta_{2^3}[1,4,2,7,6,8,7,7,1,3,1,2,1,5,7,6,1,3,1,4,1,5,8,8,3,4,2,8,6,8,8,4]$,
$H = \delta_4[1,2,1,3,1,2,3,3]$, $x \in \Delta_8$, $u \in \Delta_4$ and $y \in \Delta_4$. Let $L = [L_1, L_2, L_3, L_4]$ and $y = \delta_4^1, \delta_4^2, \delta_4^3$ represent red, gray, and white respectively. Then, the vertex-colored STG $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$ are constructed as shown in Fig. 12.



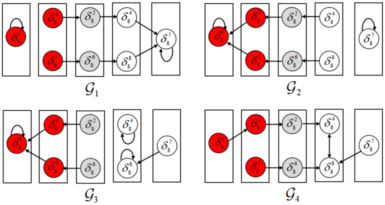



Figure 14: Vertex-colored STG $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$.

The coarsest common CP-VP of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$ is $\mathcal{S} = \{S_l\}_{l=1}^5$ with

$$S_1 = \{1\}, S_2 = \{3,5\}, S_3 = \{2,6\}, S_4 = \{4,8\}, S_5 = \{7\}.$$

Then,

$$\Delta_8/\sim = \mathcal{S} = \{S_l\}_{l=1}^5 := \{\tilde{\delta}_8^1, \tilde{\delta}_8^3, \tilde{\delta}_8^2, \tilde{\delta}_8^4, \tilde{\delta}_8^7\} \tag{73}$$

and $|\Delta_8/\sim| = 5$. Define $\phi : \Delta_8/\sim \mapsto \Delta_5$ as $\phi(S_l) = \delta_5^l, l = 1,2,3,4,5$. For any $\delta_8^i \in S_l$, define $\sim(\delta_8^i) = S_l$. Then, the quotient mapping is $\phi\circ\sim: \Delta_8/\sim \mapsto \Delta_5$.

Given that $\mathrm{Col}(H) = \{\delta_4^1, \delta_4^2, \delta_4^3\}$, then $|\mathrm{Col}(H)| = 3$.

Thus, $\varphi : y \in \Delta_4 \mapsto \tilde{y} \in \Delta_3$ can be defined as $\varphi(\delta_4^i) = \delta_3^i, \ i = 1, 2, 3$.

Define $z = \phi \circ \sim (x)$ and $\tilde{y} = \varphi(y)$.

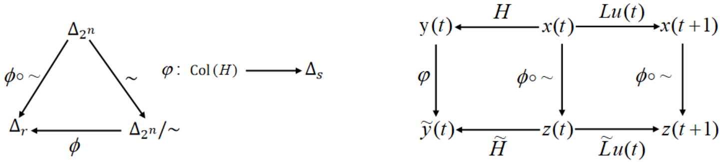



Figure 15:

Then, Omitting the computational processes, the quotient system with the algebraic form is

$$\begin{aligned} z(t+1) &= \tilde{L}u(t)z(t), \\ y(t) &= \tilde{H}z(t), \end{aligned} \tag{74}$$

with $\tilde{L} = \delta_5[1, 3, 4, 5, 5, 1, 1, 2, 3, 5, 1, 1, 2, 4, 4, 2, 3, 4, 4, 4]$, $\tilde{H} = \delta_3[1, 1, 2, 3, 3]$. The quotient system (74) is the minimum realization of the BCN (72). The vertex-colored STG of the quotient system is shown in Fig. 13.

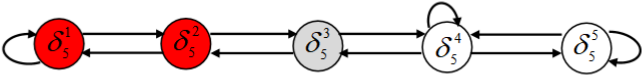



Figure 16: Vertex-colored STG of the quotient system (74).

# Some results on the realization of BCNs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# Outline

# Conclusions

1. This report mainly introduces the definition, main results, and related research progress of Boolean network decoupling.
2. Breaking through the regularity constraints and proposing some new decoupling definitions (disturbance decoupling, blocking decoupling, Kalman decomposition, realization.)
3. Proposing the vertex partition method to solve the decoupling problem of BNs (disturbance decoupling, input-output decoupling, blocking decoupling, Kalman decomposition, realization).
4. Establishing some new decoupling conditions, including algebraic and graphic conditions, using the state space method and vertex partition method.

# Open problems

(1) How to design controllers for the decoupling problems of BNs?

(2) How to solve the decoupling problems of large-scale BNs?

# Some results from our team's research on decoupling


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*Thanks!*

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