From STP to Hyper-Algebra (Part-1)
从矩阵半张量积到超代数(第一部分)

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Outline

I. STP/STA with Algebraic Structure of Matrices

☞ Algebra and Matrix in Ancient China

- 吴文俊[1] : 代数无疑是中国古代数学最发达的部分.
- Katz [2] : The idea of a matrix has a long history, dated at least from the use by Chinese scholars of the Han period for solving systems of linear equations. （矩阵历史久远, 至少可追溯到中国汉代, 用于用线性方程组)
- Crilly [3] : The matrix was initiated from 200 BC, Chinese mathematicians used it. (矩阵起源于公元前200年, 中国数学家使用了数字阵列)
- 李文林[4] : ≪ 九章算术≫ 中解线性方程组的方法就是高斯消去法.

☞ The Goal of STP

It is our responsibility to develop matrix theory and its related algebraic method, initiated by our ancestry!

- [1] D. Lin, W. Li, Y. Yu, Mathematics and Mathematics-Mechamization, Shandong Educational Press, Jinan, 2001.
- [2] V.J. Katz, A History of Mathematics, Brief Version, Springer, New York, 2004.
- [3] T. Crilly, The 50 Mathematical Problems You Must Know, (translated by Y. Wang), People's Post Elec. Press, Beijing 2012.
- [4] W. Li, A History of Mathematics, CHEP and Springer，Beijing，2000.

☞ Group

Definition 1.1

Consider a set G with an operator $*: G \to G$: $(G, *)$. It is called

(i) Semi-Group: if

$$a * (b * c) = (a * b) * c, \quad a, b, c \in G. \tag{1}$$

(ii) Monoid (semi-group with identity): if in addition to (i), there exists an identity $e \in G$, such that

$$a * e = e * a = a, \quad \forall a \in G. \tag{2}$$

Definition 1.1(cont'd)

(iii) Group: if in addition to (i) and (ii), for each $a \in G$ there exists a unique inverse a^{-1} such that

$$a * a^{-1} = a^{-1} * a = e, \quad \forall a \in G. \tag{3}$$

In addition, if

$$a * b = b * a, \quad \forall a, b \in G, \tag{4}$$

$(G, *)$ is called an abelian group.

☞ STA vs Additive Group

Example 1.2

(i) $\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, $(\mathbb{C}, +)$ are groups.

(ii) $(\mathbb{R}^n, +)$ is a group.

(iii) $(\mathcal{M}_{m\times n}, +)$ is a group.

Definition 1.3

(i)

$$\mathbb{R}^\infty = \bigcup_{n=1}^{\infty} \mathbb{R}^n.$$

(ii) Let $x, y \in \mathbb{R}^\infty$. Say, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $t = \mathrm{lcm}(m, n)$. Then

$$x \vec{\pm} y := \left(x \otimes \mathbf{1}_{t/m}\right) \pm \left(y \otimes \mathbf{1}_{t/n}\right) \in \mathbb{R}^t \subset \mathbb{R}^\infty.$$

Consider $(\mathbb{R}^\infty, \vec{+})$.
Is it a group? No!

$$\mathbf{e} = \{0_n \in \mathbb{R}^n \mid n = 1, 2, \cdots\}.$$

Hyper group: “Group “with multi-identity.

Then we might say $(\mathbb{R}^\infty, \vec{+})$ is a hyper group.

Definition 1.4

(i) Let $\mathbb{Q}^+$ be the set of positive rational numbers. $\mu = \mu_y/\mu_x \in \mathbb{Q}^+$. $\gcd(\mu_y, \mu_x) = 1$.

$$\mathcal{M}_\mu := \{A \in \mathcal{M}_{m\times n} \mid m/n = \mu\}.$$

Particularly, $\mathcal{M}_1$ is the set of square matrices.

(ii) Assume $A, B \in \mathcal{M}_\mu$. Say, $A \in \mathcal{M}_{m\mu_y \times m\mu_x}$, and $B \in \mathcal{M}_{n\mu_y \times n\mu_x}$, and $\operatorname{lcm}(m, n) = t$. Then

$$A \bar{+} B := (A \otimes I_{t/m}) + (B \otimes I_{t/n}). \tag{5}$$

$$A \vec{+} B := (A \otimes J_{t/m}) + (B \otimes J_{t/n}), \tag{6}$$

where $J_k = \frac{1}{k}\mathbf{1}_{k\times k}$.

Example 1.5

(i) $(\mathcal{M}_{m\times n}, +)$ is an abelian group.

(ii) Neither $(\mathcal{M}_\mu, \bar{+})$ nor $(\mathcal{M}_\mu, \vec{+})$ is a group.
Both $(\mathcal{M}_\mu, \bar{+})$ and $(\mathcal{M}_\mu, \vec{+})$ are abelian Hyper groups.

Definition 1.6

(i) Let $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, and $t = \mathrm{lcm}(n,p)$. The STP of A and B is

$$A \ltimes B := \left(A \otimes I_{t/n}\right)\left(B \otimes I_{t/p}\right) \in \mathcal{M}_{mt/n \times qt/p}. \tag{7}$$

(ii)

$$\mathcal{M} := \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\mathcal{M}_{m\times n}.$$

(iii)

$$\mathcal{T}_n := \{A \in \mathcal{M}_{n\times n} \mid A \text{ is invertible.}\}.$$

(iv)

$$\mathcal{T} := \bigcup_{n=1}^{\infty}\mathcal{T}_n.$$

Example 1.7

(i) $(\mathcal{M}, \ltimes)$ is a monoid with identity $e = 1$.

(ii) $(\mathcal{T}_n, \times)$ is a group.

(iii) $(\mathcal{T}, \ltimes)$ is not a group.
$(\mathcal{T}, \ltimes)$ is a hyper group.

Definition 1.8

Consider $\mathcal{M}$.

(i) Define

$$\mathcal{E}_{m\times n} := \frac{1}{\sqrt{mn}}\mathbf{1}_{m\times n}, \quad m, n \in \mathbb{Z}^+.$$

(ii) Let $A, B \in \mathcal{M}$, say $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, $\operatorname{lcm}(m,p) = s$, and $\operatorname{lcm}(n,q) = t$. Define

$$A \vec{\pm} B := (A \otimes \mathcal{E}_{s/m\times t/n}) \pm (B \otimes \mathcal{E}_{s/p\times t/q}) \in \mathcal{M}_{s\times t}. \quad (8)$$

Example 1.9

$(\mathcal{M}, \vec{+})$ is an abelian hyper group.

☞ DK-STP and Pseudo DK-STP

Definition 1.10

Let $A, B \in \mathcal{M}$. Say $A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$. Define

(i) (DK-STP)

$$A \bar{\times} B := (A \otimes \mathcal{E}^T_{t/n})(B \otimes \mathcal{E}_{t/p}) \in \mathcal{M}_{mt/n\times qt/p}, \qquad (9)$$

where $t = \mathrm{lcm}(n, p)$.

(ii) (Pseudo-DK-STP)

$$A \vec{\bar{\times}} B := (A \otimes \mathcal{E}_{s/m\times t/n}) \bar{\times} (B \otimes \mathcal{E}_{s/p\times t/q}) \in \mathcal{M}_{s\times t}, \qquad (10)$$

where $s = \mathrm{lcm}(m, p)$, and $t = \mathrm{lcm}(n, q)$.

Proposition 1.11

(i) $(\mathcal{M}, \overline{\times})$ is a semi-group.

(ii) $(\mathcal{M}, \vec{\overline{\times}})$ is a semi-group.

Poincaré: “All of mathematics is a tale about group.”
(庞加莱: 所有的数学都是群的故事.)

[5] I. James, *Remarkable Mathematicians – From Euler to von Neumann*, Cambridge Univ. Press, 2002.

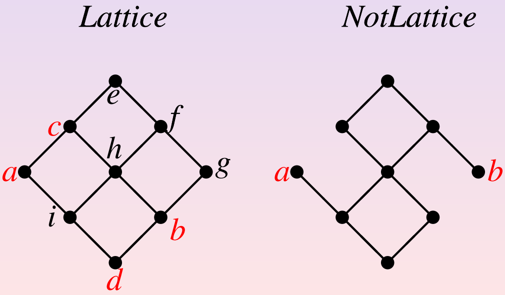
II. Hyper Group

☞ Lattice

Definition 2.1

A partial order set Λ is a lattice, if $\lambda,\ \mu \in \Lambda$, there are $\sup(\lambda, \mu)$ (or $\lambda \vee \mu$), and $\inf(\lambda, \mu)$ (or $\lambda \wedge \mu$).

Hasse diagram：in Figure 1 the left is a lattice, the right is not a littice.



Example 2.1

(i) Consider $\mathbb{Z}_+$. Set $a \prec b$ if $a \neq b$ and $a|b$, i.e. a is a proper factor of b. It follows that

$$a \vee b = \operatorname{lcm}(a, b); \quad a \wedge b = \gcd(a, b).$$

Then $(\mathbb{Z}_+, \prec)$ is a lattice, called MD-1 lattice.

(ii) Consider $\mathbb{Z}_+ \times \mathbb{Z}_+$. Set $(a, c) \prec (b, d)$ if both $a \prec b$ and $c \prec d$ (defined as in (i)). It follows that

$$(a, c) \vee (b, d) = (\operatorname{lcm}(a, b), \operatorname{lcm}(c, d));$$
$$(a, c) \wedge (b, d) = (\gcd(a, b), \gcd(c, d)).$$

Then $(\mathbb{Z}_+ \times \mathbb{Z}_+, \prec)$ is a lattice, called the MD-2 lattice.

☞ Set of Identities (with Lattice Structure)

Definition 2.2

Consider a semi-group $\mathcal{G} = (G, *)$. $\mathbf{e} \subset G$ is called an identity set of $\mathcal{G}$, if

(i)

$$g * e = e * g, \quad \forall g \in G, \ \forall e \in \mathbf{e}. \tag{11}$$

(ii) There is a lattice Λ as the index set of $\mathbf{e}$, that is,

$$\mathbf{e} = \{e_\lambda \mid \lambda \in \Lambda\}.$$

Moreover,

$$e_\lambda * e_\mu = e_{\lambda \vee \mu}, \quad \lambda, \mu \in \Lambda. \tag{12}$$

Definition 2.2(cont'd)

(iii) For each $g \in G$ there exists a unique $e_g = e_{\lambda_g} \in \mathbf{e}$ such that

$$e_\lambda * g = g, \tag{13}$$

if and only if, $\lambda \prec \lambda_g$, (including $\lambda = \lambda_g$).

☞ Hyper Group

Definition 2.3

Consider a semi-group $\mathcal{G} = (G, *)$.

(i) $\mathcal{G} = (G, *)$ is a hyper-monoid, if there exists an identity set $\mathbf{e} \subset G$.

(ii) A hyper-Monoid $\mathcal{G} = (G, *)$ is a hyper group, if for each $g \in G$, there exists a $g^{-1} \in G$ such that

$$g * g^{-1} = g^{-1} * g = e_g. \tag{14}$$

In addition, if $a * b = b * a, \quad \forall a, b \in G$, then it is called an abelian hyper-monoid/hyper group.

Example 2.4

(i) $(\mathbb{R}^\infty, \vec{+})$ is a hyper group, where

$$\mathbf{e} = \{0_n \in \mathbb{R}^n \mid n \in \mathbb{Z}_+\}, \tag{15}$$

w.r.t. MD-1 lattice.

(ii) $(\mathcal{M}_1, \bar{+})$ (or $(\mathcal{M}_1, \vec{+})$) is a hyper group, where the identity set is

$$\mathbf{e} = \{0_{n\times n} \in \mathcal{M}_{n\times n} \mid n \in \mathbb{Z}_+\}, \tag{16}$$

w.r.t. MD-1 lattice.

(iii) $(\mathcal{M}_\mu, \bar{+}))$ (or $(\mathcal{M}_\mu, \vec{+})$) is a hyper group, where the identity set is

$$\mathbf{e} = \{0_{n\mu_y\times n\mu_x} \in \mathcal{M}_{n\mu_y\times n\mu_x} \mid n \in \mathbb{Z}_+\}, \tag{17}$$

w.r.t. MD-1 lattice.

Example 2.4(cont'd)

(iv) $(\mathcal{M}, \vec{+})$ is a hyper group, where the identity set is

$$\mathbf{e} = \{0_{m\times n} \in \mathcal{M}_{m\times n} \mid (m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+\}, \tag{18}$$

w.r.t. MD-2 lattice.

Example 2.5

Recall that

$$\mathcal{T} := \bigcup_{i=1}^{\infty} \mathcal{T}_n,$$

where $\mathcal{T}_n$ is the set of $n \times n$ invertible matrices. Then $(\mathcal{T}, \ltimes)$ is a hyper group, where the identity set is

$$\mathbf{e} = \{I_n \in \mathcal{M}_{n\times n} \mid n \in \mathbb{Z}_+\}, \tag{19}$$

w.r.t. MD-1 lattice.

Proposition 2.6

Assume $\mathcal{G} = (G, *, \mathbf{e})$ is a hyper group, where $\mathbf{e}$ w.r.t. Λ. For each $e_\lambda \in \mathbf{e}$, $\lambda \in \Lambda$ set $G_\lambda \subset G$ as

$$G_\lambda := \{x \in G \mid e_x = e_\lambda\}. \tag{20}$$

Then

(i) For each $\lambda \in \Lambda$, $\mathcal{G}_\lambda = (G_\lambda, *)$ is a group, called the component group of $\mathcal{G}$, and its identity is e_λ.

(ii)

$$G = \bigcup_{\lambda \in \Lambda} G_\lambda \tag{21}$$

is a partition.

(iii) If $x \in G_\lambda$ and $y \in G_\mu$, then $x * y \in G_{\lambda \vee \mu}$.

☞ Equivalence

Definition 2.7

Consider a hyper group $\mathcal{G} = (G, *, \mathbf{e})$. Let $x,\ y \in G$. x and y are said to be equivalent, denoted by $x \sim y$, if there exist e_α and e_β such that

$$x * e_\alpha = y * e_\beta. \tag{22}$$

The equivalence class of x, denoted by $\bar{x}$, is

$$\bar{x} := \{y \in G \mid y \sim x\}.$$

Definition 2.8

Consider a hyper group $\mathcal{G} = (G, *)$. The equivalence $\sim$ is said to be consistent with the group operator $*$, if the followings are satisfied.

Definition 2.8(cont'd)

(i) If $x_1 \sim x_2$ and $y_1 \sim y_2$, then

$$x_1 * y_1 \sim x_2 * y_2. \tag{23}$$

(ii) If there exists $e \in \mathbf{e}$ such that $x \sim e$, then $x \in \mathbf{e}$.

Then we have the identity set as one element.

Proposition 2.9

Consider a hyper group $\mathcal{G} = (G, *, \mathbf{e})$. If the equivalence is consistent with the operator, then

$$\bar{e} = \mathbf{e}, \quad e \in \mathbf{e}. \tag{24}$$

Proposition 2.10

Consider a hyper group $\mathcal{G} = (G, *, \mathbf{e})$. Assume the equivalence $\sim$ defined by (22) is consistent with $*$. Then

$$\bar{x} * \bar{y} := \overline{x * y}, \tag{25}$$

is properly defined. Moreover, $\bar{\mathcal{G}} = (\bar{G}, *)$ is a group, called the equivalence group of $\mathcal{G}$, denoted by

$$\bar{\mathcal{G}} := \mathcal{G}/\sim = (\bar{G}, *),$$

where $\bar{G} = \{\bar{x} \mid x \in G\}$.

Observing Propositions 2.6 and 2.10, a geometric picture for the structure of a hyper group, called the group decomposition of hyper group, is depicted by Figure 2.

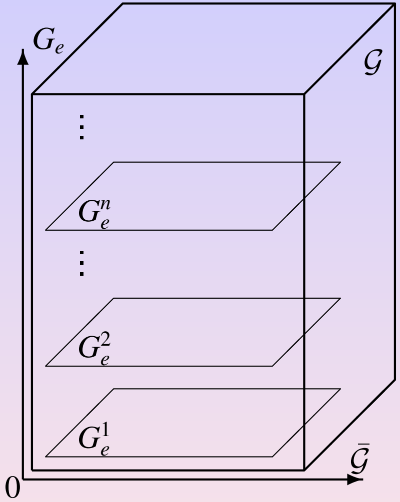


Figure 2: Group Decomposition of a Hyper-Group $\mathcal{G}$

III. Permutation Hyper Group

☞ Left Permutation Hyper-Group

$$\mathbf{S} := \bigcup_{n=1}^{\infty} \mathbf{S}_n.$$

Definition 3.1

Assume $m|n$, and $km = n$, define $\varphi_n^m : \mathbf{S}_m \to \mathbf{S}_n$ as

$$\begin{array}{c} \varphi_n^m(\sigma)((i-1)k+s) := (\sigma(i)-1)k+s, \\ \sigma \in \mathbf{S}_m, \ s \in [1,k], \ i \in [1,m]. \end{array} \tag{26}$$

Note that

$$M_{\varphi_n^m(\sigma)} = M_\sigma \otimes I_k. \tag{27}$$

Definition 3.2

Let $\sigma \in \mathbf{S}_m$, $\mu \in \mathbf{S}_n$, and $t = \mathrm{lcm}(\sigma, \mu)$. Then

$$\sigma \odot \mu = \varphi_t^m(\sigma) \circ \varphi_t^n(\mu) \in \mathbf{S}_t. \tag{28}$$

Proposition 3.3

$(\mathbf{S}, \odot, \mathbf{e})$ is a hyper group, where

$$\mathbf{e} = \{Id_n \mid n \in Z^+\}$$

w.r.t. MD-1 lattice.

Q1: What is the quotient group?

Q2: Is left permutation hyper group isomorphic to right permutation hyper group?

IV. Hyper Ring

☞ Matrix Ring

Definition 4.1

Let R be a set with $+$ and $\times$. It is a ring if

(i) $(R, +)$ is an abelian group.

(ii) $(R, \times)$ is a semi-group.

(iii) (Distributive Law)

$$\begin{aligned} (a+b)\times c &= a\times c + b\times c, \\ a\times(b+c) &= a\times b + a\times c. \quad a,b,c\in R. \end{aligned} \tag{29}$$

In addition if $(R, \times)$ is a monoid (i.e., with identity), R is called a ring with identity; if $a\times b = b\times a\ \forall a, b\in R$, R is called a commutative ring.

Example 4.2

(i) $(\mathcal{M}_{n\times n}, +, \times)$ is a ring with identity.

(ii) $(\mathcal{M}_{m\times n}, +, \overline{\times})$ is a ring.

Definition 4.3

Let R be a set with $+$ and $\times$. It is a hyper ring if

- $(R, +)$ is a hyper group.
- (ii) and (iii) are the same as in Definition 4.1.

Example 4.4

(i) $(\mathcal{M}_1, \bar{+}, \ltimes)$ (or $(\mathcal{M}_1, \vec{+}, \circ)$) is a hyper ring, where

$$A \bar{+} B := (A \otimes I_{t/m}(B \otimes I_{t/n}), \quad A \in \mathcal{M}_{m\times m}, B \in \mathcal{M}_{n\times n}.$$

$$A \vec{+} B := (A \otimes J_{t/m}(B \otimes J_{t/n}).$$

$$A \circ B = (A \otimes J_{t/m})(B \otimes J_{t/n}), \quad (J_k = \mathcal{E}_{k\times k} = \frac{1}{k}\mathbf{1}_{k\times k}).$$

(ii) $(\mathcal{M}, \vec{+}, \vec{\times})$ is a hyper ring, where ($A \in \mathcal{M}_{m\times n}$, $B \in \mathcal{M}_{p\times q}$, $s = \mathrm{lcm}(m, p)$, and $t = \mathrm{lcm}(n, q)$),

$$A \vec{+} B = (A \otimes \mathcal{E}_{s/m\times t/n}) + (B \otimes \mathcal{E}_{s/p\times t/q}) \in \mathcal{M}_{s\times t}.$$

$$A \vec{\times} B = (A \otimes \mathcal{E}_{s/m\times t/n}) \times (B \otimes \mathcal{E}_{s/p\times t/q}) \in \mathcal{M}_{s\times t}.$$

☞ Decomposition of Hyper Ring

Proposition 4.5

Consider a hyper-ring $(R, +, \times)$ with the identity set of its addition hyper group as $\mathbf{e} = \{e_\lambda \mid \lambda \in \Lambda\}$.

(i) If the addition $+$ and the product $\times$ of R are consistent, then each $(R_\lambda, +, \times)$ is a ring, called the component ring of the hyper-ring $(R, +, \times)$.

(ii) In addition to (i), if the equivalence is consistent with the operators, then the operators over quotient space can be defined properly by

$$\begin{aligned} \bar{x} + \bar{y} &:= \overline{x + y}, \quad x, y \in R, \\ \bar{x} \times \bar{y} &:= \overline{x \times y}. \end{aligned} \tag{30}$$

Therefore, the quotient space becomes a ring, called the equivalence ring.

Similarly to hyper groups, the decomposition of hyper-rings are shown in Figure 3.

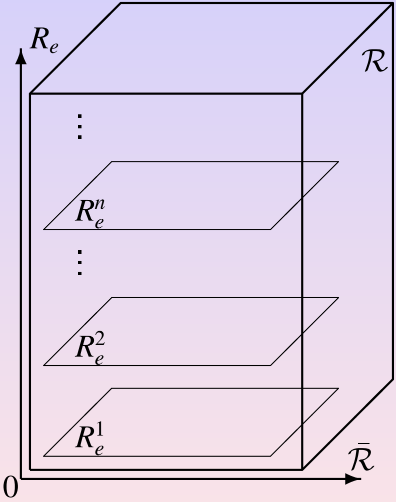


Figure 3: Ring Decomposition of a Hyper-Ring $\mathcal{R}$

V. Hyper Vector Space

☞ Vector Space

Definition 5.1 [6]

A vector space $\mathcal{V}$ over $\mathbb{R}$ is a triple $\mathcal{V} = (X, +, \cdot)$, where X is a set with elements $x \in X$ are vectors; addition $+ : X \times X \to X$, and scalar product $\cdot : \mathbb{R} \times X \to X$, satisfying

(1) $(X, +)$ is an abelian group, that is,

(i) Associativity:

$$(x + y) + z = x + (y + z), \quad x, y, z \in X. \tag{31}$$

(ii) Commutativity:

$$x + y = y + x, \quad x, y \in X. \tag{32}$$

Definition 5.1(cont'd)

(iii) Zero: There exists a unique $\vec{0} \in X$, such that

$$x + \vec{0} = \vec{0} + x = x, \quad \forall x \in X. \tag{33}$$

(iv) Inverse: For each $x \in X$ there exists a $-x \in X$, such that

$$x + (-x) = 0, \quad x \in X. \tag{34}$$

(2) Scalar product satisfies

(i) Associativity:

$$(r_1 r_2) \cdot x = r_1 \cdot (r_2 \cdot x), \quad r_1, r_2 \in \mathbb{F},\ x \in X. \tag{35}$$

Definition 5.1(cont'd)

(ii) Distributive Law:

$$\begin{aligned}(r_1 + r_2) \cdot x &= r_1 \cdot x + r_2 \cdot x, \quad r, r_1, r_2 \in \mathbb{R}, \\ r \cdot (x + y) &= r \cdot x + r \cdot y, \quad x,\ y \in X.\end{aligned} \tag{36}$$

(iii) Unit:

$$1 \cdot x = x, \quad 1 \in \mathbb{R}, x \in X. \tag{37}$$

Definition 5.2

A hyper vector space is structurally similar to a vector space except that $(X, +)$ is an abelian hyper group.

[6] W. Greub, *Linear Algebra*, 4 ed., Springer-Verlag, New York, 1981.

Example 5.3

(i) $(\mathbb{R}^\infty, \vec{+})$ is a hyper-vector space.

(ii) $(\mathcal{M}_1, \bar{+})$ is a hyper-vector space. $(\mathcal{M}_1, \vec{+})$ is also a hyper-vector space.

(iii) $(\mathcal{M}_\mu.\vec{+})$ (or $(\mathcal{M}_\mu.\bar{+})$) is a hyper-vector space.

(iv) $(\mathcal{M}, \vec{+})$ is a hyper-vector space.

Proposition 5.4

Consider a hyper-vector space $\mathcal{V} = (X, +, \cdot)$. Assume $\mathbf{e} = \{e_\lambda \mid \lambda \in \Lambda\}$. Then

(i) For each $\lambda \in \Lambda$, X_λ is a vector space.

(ii) Assume the addition is consistent w.r.t. equivalence, then the equivalence group is a vector space.

VI. Hyper Module

☞ Hyper Module

Definition 6.1

Let R be a hyper ring. A (left) R-module is an additive (abelian) hyper group A with a function $\pi : R \times A \to A$ such that

(i)

$$r(a+b) = ra + rb, \quad r \in R,\ a,\ b \in A; \tag{38}$$

(ii)

$$(r+s)a = ra + sa, \quad r, s \in R,\ a,\ b \in A; \tag{39}$$

Definition 6.1(cont'd)

(iii)

$$r(sa) = (rs)a. \tag{40}$$

If R has an identity set $\mathbf{e}_R$ for the product, such that

$$e_R x \sim x \quad e_R \in \mathbf{e}_R, \forall x \in A, \tag{41}$$

and there exists at least one $e_x \in \mathbf{e}_R$, such that

$$e_x x = x, \tag{42}$$

then A is said to be a unitary hyper R-module.

VII. Conclusion

☞ Hyper-Algebra

$$\begin{cases}\text{Hyper Group} \Rightarrow \text{Hyper Ring} \Rightarrow \text{Hyper Module} \\ \text{Hyper Vector Space}\end{cases}$$
$$\Rightarrow \text{Cross-dimensional linear (control) systems}$$

- [7] D. Cheng, *From Dimension-Free Matrix Theory to Cross-Dimensional Dynamic Systems*, Elsevier, London, 1019.
- [8] D. Cheng, Cross-Dimensional Mathematics – A Foundation for STP/STA, (preprint: http:arxiv.org/abs/2406.12920), 2024.

Thank you!

Any Question?