STABILIZATION AND RECONSTRUCTIBILITY OF BOOLEAN NETWORKS

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Outline

1. **Stabilization and Reconstruction of Sampled-Data BCNs under Noisy Sampling Interval**
2. **Stabilization of Aperiodic Sampled-Data BCNs: A Delay Approach**
3. **State Estimation of BCNs under Stochastic Disturbances with Random Delay in Measurements**

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2. Stabilization of Aperiodic Sampled-Data BCNs: A Delay Approach
3. State Estimation of BCNs under Stochastic Disturbances with Random Delay in Measurements

Motivation

- The global STABILIZATION problem for Boolean control network (BCN) is to find, if possible, $u(t)$ such that the BCN becomes globally convergent. In the case of disease treatment, one may want to design therapeutic interventions that steer the patient to the desirable state, such as the healthy one, and maintain this state afterwards.
- In recent years, the artificial control technology of gene expression has attracted extensive attention. Akutsu *et al*. [1] stated that "one of the major goals of systems biology is to develop a control scheme for complex biological systems".
- It is worth noting that in practical applications, the system state is not always available, so researchers often consider SAMPLED-DATA CONTROL, which can be described as follows:

$$u(t) = Ex(t_k), t_k \leq t < t_{k+1},$$

where $t_{k+1} - t_k$ are sampling intervals.

[1] T. Akutsu, M. Hayashida, W. K. Ching and M. K. Ng, "Control of Boolean networks: Hardness results and algorithms for tree structured networks," *Journal of Theoretical Biology*, vol. 244, no. 4, pp. 670–679, 2007.

- As Ballesta *et al.* mentioned in [2], the effectiveness of drugs is affected by the time of administration. The administration time can be best coordinated with the daily rhythm. The daily rhythm is produced by an endogenous timing mechanism, which is very sensitive to external signals. The prediction of circadian time by machine learning and mathematical model was proposed in [3], but the prediction also has errors. Here, since drug treatment can be regarded as treatment input, we use NOISY SAMPLING INTERVAL to represent these errors.
- **In order to characterize these errors and study whether the uncertainty in the implementation of the controller leads to the instability of the system**, the noisy sampling interval is proposed and stabilization for sampled-data BCNs under noisy sampling interval is investigated.

[2] A. Ballesta, P. F. Innominato, R. Dallmann, *et al.*, "Systems chronotherapeutics," *Pharmacological Reviews*, vol. 69, no. 2, pp. 161–199, 2017.

[3] J. Hesse, D. Malhan, M. Yalçin, *et al.*, "An optimal time for treatment-predicting circadian time by machine learning and mathematical modelling," *Cancers*, vol. 12, no. 11, pp. 3103, 2020.

Consider the following BCN and sampled-data control:

$$x(t+1) = Lu(t)x(t), \tag{1}$$

$$u(t) = Kx(t_k),\ \ t_k \le t < t_{k+1}, \tag{2}$$

where $L \in \mathcal{L}_{2^n \times 2^{n+m}}$ and $K \in \mathcal{L}_{2^m \times 2^n}$.
By substituting (2) into (1), one can obtain

$$\begin{cases} x(t_k+1) = Lu(t_k)x(t_k) = LKW_{[2^n]}\Phi_n x(t_k), \\ x(t_k+2) = (LKW_{[2^n]})^2\Phi_n^2 x(t_k), \\ \vdots \\ x(t_{k+1}) = (LKW_{[2^n]})^{t_{k+1}-t_k}\Phi_n^{t_{k+1}-t_k} x(t_k). \end{cases}$$

Let $T_k = t_{k+1} - t_k$ be the sampling interval and denote $x(t_k)$ by $x[k]$. We have a system of the following form:

$$x[k+1] = (LKW_{[2^n]})^{T_k}\Phi_n^{T_k} x[k]. \tag{3}$$

Problem Statement

Notably, the sampling interval considered here is disturbed by noise and we describe the noisy sampling interval as follows:

$$T_k = T + v_k,$$

where T is a constant and v_k is a random variable. Here, the following two types of v_k are considered.

1. the probability of v_k choosing i is p_i;
2. the random variable v_k follows a discrete-time homogeneous Markov chain.

Definition 1.1

Given $x^* \in \Delta_{2^n}$, sampled-data BCN (1) under noisy sampling interval is said to be globally stochastically stable at x^*, if

$$\lim_{t\to\infty} \mathbb{E}\{x(t)|x(0) = x_0\} = x^*, \ \forall x_0 \in \Delta_{2^n}.$$

Noisy sampling interval satisfies a given probability distribution

First, we assume that v_k takes values from the set $\mathcal{M} = \{-T+1, -T+2, \ldots, l\}$, $l \in [0 : 2^n - T]$, which means that $T_k \in [1 : T + l]$. The probability of v_k choosing $i \in [-T + 1 : l]$ is $P(v_k = i) = p_i \geq 0$.

Then sampled-data BCN (1) under noisy sampling interval is transformed into the following special probabilistic Boolean network (PBN):

$$x[k + 1] = Ax[k], \tag{4}$$

where $A \in \mathcal{L}_{2^n \times 2^n}$ is chosen from the set $\{A_{-T+1}, A_{-T+2}, \ldots, A_l\}$,

$$A_i = (LKW_{[2^n]})^{T+i} \Phi_n^{T+i}, i \in [-T + 1 : l]$$

and $P(A = A_i) = P(v_k = i) = p_i$. Let $x_0 = x(0) = x[0]$, where $t_0 = 0$.

For large-scale sampled-data BCNs, we consider the following transformation.

Now, for each A_i, let $\{\hat{\alpha}_1^i, \hat{\alpha}_2^i, \dots, \hat{\alpha}_{r(i)}^i\}$ be the set of distinct indices in the set $\{\alpha_1^{(T+i)}, \alpha_2^{(T+i)}, \dots, \alpha_{2^n}^{(T+i)}\}$, where $\hat{\alpha}_1^i < \hat{\alpha}_2^i < \cdots < \hat{\alpha}_{r(i)}^i$ and let s_j^i be the number of indices in $\{\alpha_1^{(T+i)}, \alpha_2^{(T+i)}, \dots, \alpha_{2^n}^{(T+i)}\}$ coinciding with $\hat{\alpha}_j^i$, $j \in [1 : r(i)]$. Using a permutation matrix $Q_i \in \mathcal{L}_{2^n \times 2^n}$, we have

$$A_i Q_i = \delta_{2^n}[\underbrace{\hat{\alpha}_1^i \ \cdots \ \hat{\alpha}_1^i}_{s_1^i} \ \underbrace{\hat{\alpha}_2^i \ \cdots \ \hat{\alpha}_2^i}_{s_2^i} \ \cdots \ \underbrace{\hat{\alpha}_{r(i)}^i \ \cdots \ \hat{\alpha}_{r(i)}^i}_{s_{r(i)}^i}].$$

Then we can factorize A_i as follows: $A_i = A_i^1 A_i^2$, where $A_i^1 \in \mathcal{M}_{2^n \times r}$, $A_i^2 \in \mathcal{L}_{r \times 2^n}$ and

$$A_i^1 = \delta_{2^n}[\underbrace{0 \ \cdots \ 0}_{j_1^i - 1} \ \hat{\alpha}_1^i \ \underbrace{0 \ \cdots \ 0}_{j_2^i - j_1^i - 1} \ \hat{\alpha}_2^i \ \cdots \ \hat{\alpha}_{r(i)}^i \ \underbrace{0 \ \cdots \ 0}_{r - j_{r(1)}^i}]$$

$$A_i^2 = \delta_r[\underbrace{j_1^i \ \cdots \ j_1^i}_{s_1^i} \ \underbrace{j_2^i \ \cdots \ j_2^i}_{s_2^i} \ \cdots \ \underbrace{j_{r(1)}^i \ \cdots \ j_{r(1)}^i}_{s_{r(1)}^i}] Q_i^{-1}.$$

Now, we define a bijective map from $\{\delta_{2^n}^{\alpha_j} : \alpha_j \in \Gamma\}$ to Δ_r as $\varphi(\delta_{2^n}^{\alpha_j}) = \delta_r^j,\ \forall j \in [1 : r]$. Setting $\hat{A}_i = A_i^2 A_i^1 \in \mathcal{M}_{r\times r}$, we obtain a new probabilistic logic network as follows:

$$z[k+1] = \hat{A}_i z[k] \tag{5}$$

where $z[k] \in \Delta_r \cup \{\delta_r^0\}$ and $i \in [-T+1 : l]$.

Theorem 1.2

Consider sampled-data BCN (1) under noisy sampling interval. If the following conditions hold

1. $[\hat{A}_{-T+1}]_{j,j} = 1$;
2. *there exists* $\hat{k} \in [1 : r-1]$, *such that* $Row_j[(\hat{\mathrm{A}} \ltimes \mathrm{p})^{\hat{k}}] > 0$,

where $\hat{\mathrm{A}} = [\hat{A}_{-T+1}\ \hat{A}_{-T+2}\ \cdots\ \hat{A}_l]$, $\hat{A}_i = A_i^2 A_i^1, i \in [-T+1 : l]$ *and* $\mathrm{p} = [p_{-T+1}\ p_{-T+2}\ \cdots\ p_l]^{\mathrm{T}}$, $P(A = A_i) = P(v_k = i) = p_i$, *then sampled-data BCN (1) under noisy sampling interval is globally stochastically stable at* $\delta_{2^n}^{\alpha_j}$.

Noisy sampling interval follows a discrete-time homogeneous Markov chain

Assume that v_k ($k \in \mathbb{N}$) follows a discrete-time homogeneous Markov chain taking values in a finite set $\mathcal{M} = \{-T+1, -T+2, \ldots, l\}$ with the transition probability matrix $\prod = [\pi_{ab}]$ where

$$\pi_{ab} = P(v_{k+1} = b | v_k = a), \tag{6}$$

$\pi_{ab} \geq 0$ for $a, b \in \mathcal{M}$, and $\Sigma_{b=-T+1}^{l} \pi_{ab} = 1$ for $a \in \mathcal{M}$.

Then sampled-data BCN (1) under noisy sampling interval can be transformed into the following stochastic BN

$$x[k+1] = A_{v_k} x[k], \tag{7}$$

where $A_i = (LKW_{[2^n]})^{T+i} \Phi_n^{T+i},\ i \in [-T+1 : l]$.

Corollary 1.3

Consider sampled-data BCN (1) under noisy sampling interval. If there exists vector $\hat{\lambda}_b \in \mathbb{R}^r,\ b \in \mathcal{M}$ such that the following conditions hold

1. $\hat{\lambda}_b^{\mathsf{T}} \delta_r^j = 0$;
2. $\hat{\lambda}_b^{\mathsf{T}} \delta_r^k > 0$;
3. $[\hat{A}_{-T+1}]_{j,j} = 1$;
4. $(\Sigma_{b=-T+1}^{l} \pi_{ab} \hat{\lambda}_b^{\mathsf{T}} \hat{A}_a - \hat{\lambda}_a^{\mathsf{T}}) \delta_r^k < 0$,

for $a \in [-T+1 : l],\ k \in [1 : r]$ and $j \neq k$, and $\hat{A}_i = A_i^2 A_i^1$ $(A_i = A_i^1 A_i^2),\ i \in [-T+1 : l]$, then sampled-data BCN (1) under noisy sampling interval is globally stochastically stable at $\delta_{2^n}^{\alpha_j}$.

Remark 1.4

By transforming the considered PBN (4) (or (7)) into a size-reduced probabilistic logic network $z[k+1] = \hat{A}_i z[k],\ i \in [-T+1 : l]$, where the dimension of PBN is 2^n and the dimension of logical probabilistic network is r, the above Theorem 1.2 and Corollary 1.3 are applicable to large-scale sampled-data BCNs.

Reconstruction of sampled-data BCNs under noisy sampling interval

Assume that v_k takes values from $\{-T+1, -T+2, \ldots, -T+N\}$, $N \in [1:2^n]$, which means that $T_k \in [1:N]$. The probability of v_k choosing $i \in [-T+1 : -T+N]$ is $P(v_k = i) \geq 0$.
Regard

$$x_i[k+1] = A_0^{(i)} x[k], \tag{8}$$

where $A_0^{(i)} = M_i K W_{[2^n]} (LKW_{[2^n]})^{T-1} \Phi_n^T$ as the main dynamics and then regard

$$x_i[k+1] = A_l^{(i)} x[k], \tag{9}$$

where $A_l^{(i)} = M_i K W_{[2^n]} (LKW_{[2^n]})^{l-1} \Phi_n^l,\ l \neq T,\ l \in [1:N]$ as the noisy dynamics.

Let p_0 be the probability that $A_0^{(i)}$ is selected and p_l be the probability that $A_l^{(i)}$ is selected, where $l \in [1:N]$ and $l \neq T$. Let $x[0] = x_0$ and $Q = \{l : l \neq T, l \in [1:N]\}$. Let the steady-state probability s_i, $i \in [1:n]$ satisfy $s_i = \mathbb{E}\{x_i[k+1]|x_0\} = \mathbb{E}\{x_i[k]|x_0\}$. The reconstruction problem of sampled-data BCN under noisy sampling interval can be formulated as follows:

Problem 1.5

Consider a sampled-data BCN under noisy sampling interval. Suppose that the main dynamics (8) and the steady-state probabilities s_i, $i \in [1:n]$ are given. Then we are to find

1. *probabilities p_0 and p_l, $l \in Q$;*
2. *matrices $A_l^{(i)}$, $l \in Q$, $i \in [1:n]$*

maximizing p_0 subject to $p_0 + \sum_{l\in Q} p_l = 1$, $A_l^{(i)} = M_i K W_{[2^n]}(LKW_{[2^n]})^{l-1}\Phi_n^l$, $l \in Q$ and

$$s_i = (p_0 A_0^{(i)} + \sum_{l\in Q} p_l A_l^{(i)})s, \ \forall i \in [1:n],$$

where $s = \ltimes_{i=1}^n s_i$.

The above problem can be formulated as a linear programming problem:

$$
\begin{aligned}
& \max \; p_0 \\
\text{s.t. } & \bar{s}_i = p_0 \sum_{j \in \mathcal{I}(A_0^{(i)})} [s]_j + \sum_{l \in Q} p_l \sum_{j \in \mathcal{I}(A_l^{(i)})} [s]_j, \text{ for } i \in [1:n], \\
& p_0 + \sum_{l \in Q} p_l = 1, \\
& 0 \le p_0 \le 1, \\
& 0 \le p_l \le 1, \; l \in Q, \\
& A_0^{(i)} = M_i K W_{[2^n]} (L K W_{[2^n]})^{T-1} \Phi_n^T, \; i \in [1:n], \\
& A_l^{(i)} = M_i K W_{[2^n]} (L K W_{[2^n]})^{l-1} \Phi_n^l, \; i \in [1:n], \; l \in Q.
\end{aligned}
$$

Remark 1.6

According to the solution of the above linear programming problem, sampled-data BCN (1) under noisy sampling interval can be reconstructed as follows: the noisy sampling interval is T_k, where the probability of T_k choosing T is p_0 and the probability of T_k choosing $l \in Q$ is p_l.

Conclusions

In **[S1]**, we study stabilization and reconstruction of sampled-data BCNs under noisy sampling interval. There are some contributions in this part.

1. For a large-scale sampled-data BCN, we transform it into a size-reduced probabilistic logical network. Then by studying the stochastic stability of the probabilistic logical network, some sufficient conditions for global stochastic stability of the large-scale sampled-data BCN are obtained. In addition, for general large-scale PBNs, the above transformation method is also applicable.
2. The reconstruction of BNs and PBNs is not only a fundamental but essential problem in systems biology. Based on the given steady-state probabilities of the transformed PBN, the reconstruction problem of sampled-data BCNs under noisy sampling interval can be well-solved as a linear programming problem. Notably, the reconstruction method we presented here is also applicable to large-scale sampled-data BCNs.

[S1] Liangjie Sun and Wai-Ki Ching, "Stabilization and reconstruction of sampled-data Boolean control networks under noisy sampling interval," *IEEE Transactions on Automatic Control*, vol. 68, no. 4, pp. 2444–2451, 2023.

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Motivation

1. In [4], Liu *et al.* first applied sampled-data state feedback control (SDSFC) to the stabilization problem of BCNs. It was mentioned in [5] that by comparing the traditional state feedback control with SDSFC, the number of controller updates could be reduced by using SDSFC. The sampling interval of the SDSFC is constant.
2. In fact, for real-world engineering problems, the sampling interval is usually not constant. Aperiodic sampled-data control (ASDC) is proposed, whose sampling interval is uncertain and stochastic. As mentioned in Wu *et al.* [6], the utilization of ASDC can further reduce the costs of energy, computation and communication.

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[4] Y. Liu, J. Cao, *et al.*, "Sampled-data state feedback stabilization of Boolean control networks," *Neural Computation*, vol. 28, no. 4, pp. 778–799, 2016.

[5] L. Tong, Y. Liu, F. E. Alsaadi and T. Hayat, "Robust sampled-data control invariance for Boolean control networks," *Journal of the Franklin Institute*, vol. 354, no. 15, pp. 7077–7087, 2017.

[6] Y. Wu, H. Su, *et al.*, "Consensus of multiagent systems using aperiodic sampled-data control," *IEEE Transactions on Cybernetics*, vol. 46, no. 9, pp. 2132–2143, 2015.

Motivation

- We observe that in [7,8], very often they convert sampled-data control into time-delay control. Therefore, the motivation of this part is how to transform the ASDC into delayed control and how to apply the time-delay approach to analyze the stochastic stability of BCNs under ASDC.
- The key challenge is how to ensure that the transformed delayed control is consistent with the original ASDC.

[7] Y. Xu, H. Su and Y. Pan, "Output feedback stabilization for markov-based nonuniformly sampled-data networked control systems," *Systems & Control Letters*, vol. 62, no. 8, pp. 656–663, 2013.

[8] E. Fridman, A. Seuret and J. P. Richard, "Robust sampled-data stabilization of linear systems: An input delay approach," *Automatica*, vol. 40, no. 8, pp. 1441–1446, 2004.

A BCN under ASDC can be described as follows:

$$x(k+1) = Lu(k)x(k), \tag{10}$$

$$u(k) = Ex(\theta_i),\ \theta_i \leq k < \theta_{i+1}. \tag{11}$$

where $x(k) \in \Delta_{2^n},\ u(k) \in \Delta_{2^m},\ L \in \mathcal{L}_{2^n \times 2^{n+m}}$ and $E \in \mathcal{L}_{2^m \times 2^n}$.

It is worth noting that the sampling instants $\theta_i,\ i = 1, 2, \ldots$ are uncertain. $\theta_0 = 0$. Denote $h_i \triangleq \theta_{i+1} - \theta_i$ the i-th sampling interval. Here we assume that the sampling interval is bounded above by $N+1$, i.e., $1 \leq h_i \leq N+1,\ i = 0, 1, \ldots$.

Convert a ASDC into a Delayed Control

The ASDC can be represented as a delayed control as follows:

$$u(k) = Ex(k - \tau_k),\ \theta_i \leq k < \theta_{i+1}, \tag{12}$$

where $\tau_k \triangleq k - \theta_i$ is a random variable.

- Since the value of τ_{k+1} is only related to τ_k, we assume that τ_k follows a Markov chain taking values in a finite set $\mathcal{M} = \{0, 1, \ldots, N\}$.
- When $\tau_k = j \neq N,\ j \in \mathcal{M}$, if $\theta_{i+1} - \theta_i > j + 1$, then $\theta_i \leq k + 1 < \theta_{i+1}$ and $\tau_{k+1} = k + 1 - \theta_i = \tau_k + 1 = j + 1$; if $\theta_{i+1} - \theta_i = j + 1$, then $k + 1 = \theta_{i+1}$ and $\tau_{k+1} = k + 1 - \theta_{i+1} = 0$. Specially, when $\tau_k = N$, then $\tau_{k+1} \equiv 0$ $(h_i \leq N + 1)$.

Therefore, the transition probability matrix of τ_k is $\Pi = [\pi_{ab}]$, where $\pi_{ab} = \mathbf{P}(\tau_{k+1} = b | \tau_k = a),\ a, b \in \mathcal{M}$, can be expressed as follows:

$$\Pi = \begin{bmatrix} \pi_0 & 1-\pi_0 & 0 & \cdots & 0 & 0 \\ \pi_1 & 0 & 1-\pi_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \pi_{N-1} & 0 & 0 & \cdots & 0 & 1-\pi_{N-1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where $\pi_j \in (0,1),\ j = 0, 1, \ldots, N-1$.

Global Stability

Consider BCN (10) and ASDC (11). For $\theta_i \leq k < \theta_{i+1}$, we have

$$\begin{cases} x(\theta_i+1) = LEW_{[2^n]}\Phi_n x(\theta_i), \\ x(\theta_i+2) = (LEW_{[2^n]})^2(\Phi_n)^2 x(\theta_i), \\ \quad \vdots \\ x(\theta_{i+1}) = (LEW_{[2^n]})^{\theta_{i+1}-\theta_i}(\Phi_n)^{\theta_{i+1}-\theta_i} x(\theta_i), \end{cases}$$

i.e.,

$$x(k+1) = (LEW_{[2^n]})^{k+1-\theta_i}(\Phi_n)^{k+1-\theta_i} x(\theta_i), \; \theta_i \leq k < \theta_{i+1}. \tag{13}$$

Definition 2.1

BCN (10) under ASDC (11) is said to be globally stochastically stable at $\delta_{2^n}^{2^n}$, if for any initial value $x(0) \in \Delta_{2^n}$ and $\theta_0 = 0$, the trajectory $x(k)$ of system (13) satisfies $\lim_{k\to\infty} \mathbb{E}\{x(k)|x(0), \theta_0 = 0\} = \delta_{2^n}^{2^n}$.

By $\tau_k = k - \theta_i,\ \theta_i \leq k < \theta_{i+1}$, one can get that

$$x(k+1) = (LEW_{[2^n]})^{\tau_k+1}(\Phi_n)^{\tau_k+1}x(k-\tau_k). \tag{14}$$

Consider system (14) and define the augmented state vector $\mathbf{X}(k) = [x^{\mathsf{T}}(k)\ x^{\mathsf{T}}(k-1)\ \cdots\ x^{\mathsf{T}}(k-N)]^{\mathsf{T}}$, one has

$$\mathbf{X}(k+1) = G(\tau_k)\mathbf{X}(k), \tag{15}$$

where

$$G(\tau_k) = \begin{bmatrix} A(\tau_k) \\ B \end{bmatrix} \in M_{(N+1)2^n \times (N+1)2^n},$$

and

$$B = \begin{bmatrix} I_{2^n} & 0 & \cdots & 0 & 0 \\ 0 & I_{2^n} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{2^n} & 0 \end{bmatrix} \in M_{N2^n \times (N+1)2^n}.$$

For simplicity, we define $((\delta_{2^n}^{2^n})^{\mathsf{T}}\ (\delta_{2^n}^{2^n})^{\mathsf{T}}\ \cdots\ (\delta_{2^n}^{2^n})^{\mathsf{T}})^{\mathsf{T}}$ as Y.

Lemma 2.2

BCN (10) under ASDC (11) is globally stochastically stable at $\delta_{2^n}^{2^n}$, if and only if for any $\mathbf{X}(0)$ and $\tau_0 = 0$:

$$\lim_{k\to\infty} \mathbb{E}\{\mathbf{X}(k), k|\mathbf{X}(0), \tau_0 = 0\} = Y.$$

Theorem 2.3

Consider BCN (10) under ASDC (11). If there exist vectors $0 \leq \beta(i) \in \mathbb{R}^{(N+1)2^n}$, $i \in \mathcal{M}$ and the following inequalities hold for all $i \in \mathcal{M}$,

$$\left(\sum_{j\in\{0,i+1\}} \pi_{ij}\beta^{\mathsf{T}}(j)G(j) - \beta^{\mathsf{T}}(i)\right)\mathbf{X}(k) < 0,\ \tau_{k-1} = i,$$

$$\left(\sum_{j\in\{0,i+1\}} \pi_{ij}\beta^{\mathsf{T}}(j)G(j) - \beta^{\mathsf{T}}(i)\right) Y = 0,$$

especially, when $i = N$, then $j = 0$, then BCN (10) under ASDC (11) is globally stochastically stable at $\delta_{2^n}^{2^n}$.

Conclusions

In **[S2]**, a novel method for the global stochastic stability analysis of aperiodic sampled-data BCNs is introduced. The main contributions of this part are as follows:

1. When the sampling instants are uncertain and only the activation frequencies of the sampling interval are known, by transforming the ASDC into delayed control, the global stochastic stability of the BCN under this ASDC is considered for the first time.
2. For SDSFC (constant sampling interval), we can also convert it into delayed control, and then the global stability of the BCN under SDSFC can be studied by a delay approach.

[S2] Liangjie Sun, Wai-Ki Ching and Jianquan Lu, "Stabilization of aperiodic sampled-data Boolean control networks: A delay approach," *IEEE Transactions on Automatic Control*, vol. 66, no. 11, pp. 5606-5611, 2021.

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MOTIVATION

- The key to solve some problems in control theory is to identify certain internal state of a system. Compared with input and output data, the internal state of a system is difficult to measure directly in most cases. Therefore, STATE ESTIMATION is a meaningful and challenging research topic in control theory.
- For BCNs, state estimation has been investigated from the following two aspects: observability [9,10] and RECONSTRUCTIBILITY [11,12] (also known as detectability). The former discusses whether one can confirm the initial state $x(0)$ based on input and output observations over a period of time $[0, T]$. The latter discusses whether one can determine the end state $x(T)$ based on input and output observations in the interval $[0, T]$.

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[9] D. Cheng and H. Qi, "Controllability and observability of Boolean control networks," *Automatica*, vol. 45, no. 7, pp. 1659–1667, 2009.

[10] R. Li, M. Yang and T. Chu, "Observability conditions of Boolean control networks," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 17, pp. 2711–2723, 2014.

[11] E. Fornasini and M. E. Valcher, "Observability, reconstructibility and state observers of Boolean control networks," *IEEE Transactions on Automatic Control*, vol. 58, no. 6, pp. 1390–1401, 2012.

[12] B. Wang, J. Feng, H. Li and Y. Yu, "On detectability of Boolean control networks," *Nonlinear Analysis: Hybrid Systems*, vol. 36, pp. 100859, 2020.

Motivation

- All the measurements considered above are all up-to-date, that is, the output measurements of the system are described as $y(t) = h(x(t))$ or $y(t) = h(v(t), x(t))$, where $v(t)$ represents the noise measurement.
- However, in many practical cases, that is, in control systems and real-time distributed decision-making, sensor data might be lost, received out of sequence or received with random delay, so that the available measurements are not up-to-date.
- Therefore, it is essential and meaningful to investigate the state estimation of BCNs under stochastic disturbances with random delay in measurements. Moreover, to our best knowledge, this issue is still an open and challenging problem.

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[13] Y. Guo, Q. Li and W. Gui, "Optimal state estimation of Boolean control networks with stochastic disturbances," *IEEE Transactions on Cybernetics*, vol. 50, no. 3, pp. 1355–1359, 2018.

[14] H. Chen, Z. Wang, J. Liang, and M. Li, "State estimation for stochastic time-varying Boolean networks," *IEEE Transactions on Automatic Control*, vol. 65, no. 12, pp. 5480–5487, 2020.

System Description

Consider the following BCN

$$x(t) = Lu(t-1)w(t-1)x(t-1), \tag{16}$$

and

$$y(t) = Hv(t)x(t), \tag{17}$$

where $x(t) \in \Delta_N$ is the state of BCN with initial state x_0. Here $u(t) \in \Delta_M$ stands for the control input and we consider that $\{u(t)\}$ is pre-specified, $y(t) \in \Delta_Q$ is the output at time t. Furthermore, $w(t) \in \Delta_{S_1}$ and $v(t) \in \Delta_{S_2}$ represent the process disturbance and the measurement noise with probability distribution vectors

$$\mathbb{P}\left(w(t) = \delta_{S_1}^{l}\right) = [\mathbf{q}^w]_l, \ l \in [1 : S_1],$$

and

$$\mathbb{P}\left(v(t) = \delta_{S_2}^{l}\right) = [\mathbf{q}^v]_l, \ l \in [1 : S_2],$$

respectively. Moreover, $v(t)$ and $w(t)$ are mutually independent and that they are independent of $x(0), x(1), \ldots, x(t)$ for any t.

Problem Statement

Here we consider the following two types of measurements:

1. The received measurement $y(t)$ may take one of the following forms:

$$y(t) = \phi \quad \text{or} \quad y(t) = Hv(t)x(t-i),\ i \in [0:d], \tag{18}$$

   where $y(t) = \phi$ means that the filter receives nothing at time t. In other cases, measurements are received with/without delays. The maximum time delay d is known. In addition, the probability of receiving the measurement $y(t) = Hv(t)x(t-i)$ is $\mathbb{P}(y(t) = Hv(t)x(t-i)) = \tau^i,\ i \in [0:d]$. Thus, the probability of data loss at time t is $\mathbb{P}(y(t) = \phi) = 1 - \sum_{i=0}^{d} \tau^i$.

2. The received measurement is given as below:

$$y(t) = Hv(t)x(t - d(t)), \tag{19}$$

   where $d(t)$ represents the random time delay. Moreover, $d(t)$ considered here is governed by a discrete-time Markov chain with the finite state-space $\{0, 1, \ldots, d\}$, and the transition probability matrix of $d(t)$ is $\Lambda = [\lambda_{ij}],\ i, j \in [0:d]$.

Our aim is to estimate the state of BCN (16) from the measurements with random delay.

Estimate the state of BCN (16) from measurement (18)

For any $k \geq t$, we define $\mathbf{q}^x_{k/t} \in \Psi_N$ as $\mathbf{q}^x_{k/t} = \left[\left[\mathbf{q}^x_{k/t}\right]_1 \ \cdots \ \left[\mathbf{q}^x_{k/t}\right]_N\right]^\mathsf{T}$ where

$$\left[\mathbf{q}^x_{k/t}\right]_j := \mathbb{P}\left(x(k) = \delta_N^j | y[0:t] = y_{[0:t]}\right),\ j \in [1:N].$$

For any $k \geq t$, we define $\mathbf{q}^{\bar{x}}_{k/t} \in \Psi_{N^{d+1}}$ as $\mathbf{q}^{\bar{x}}_{k/t} = \left[\left[\mathbf{q}^{\bar{x}}_{k/t}\right]_1 \ \cdots \ \left[\mathbf{q}^{\bar{x}}_{k/t}\right]_{N^{d+1}}\right]^\mathsf{T}$, where $\left[\mathbf{q}^{\bar{x}}_{k/t}\right]_j := \mathbb{P}\left(\bar{x}(k) = \left[(\delta_N^{i_0})^\mathsf{T} \ \cdots \ (\delta_N^{i_d})^\mathsf{T}\right]^\mathsf{T} | y[0:t] = y_{[0:t]}\right)$, and $\bar{x}(k) = \left[x(k)^\mathsf{T} \ \cdots \ x(k-d)^\mathsf{T}\right]^\mathsf{T}$, $x(k) = \delta_N^{i_0}, \ldots, x(k-d) = \delta_N^{i_d}$, $j = (i_0 - 1)N^d + (i_1 - 1)N^{d-1} + \cdots + i_d$. For the sake of brevity, let $\phi_N(i_0, i_1, \ldots, i_d) = (i_0 - 1)N^d + (i_1 - 1)N^{d-1} + \cdots + i_d$.

The measurement (18) can be described as

$$y(t) = \phi \quad \text{or} \quad y(t) = Hv(t)C^i\bar{x}(t),\ i \in [0:d],$$

where $C^i\bar{x}(t) = x(t-i)$, i.e., $C^0 = [I_N \ 0 \ \cdots \ 0], C^1 = [0 \ I_N \ \cdots \ 0], \ldots, C^d = [0 \ 0 \ \cdots \ I_N]$.

Theorem 3.1

Consider BCN (16) with measurement (18). Let $\mathbf{q}^{\bar{x}}_{0/-1}$ *be the initial probability distribution vector of the augmented state* $\bar{x}(t)$ *and* $\mathbf{q}^{x}_{0/-1} = \mathbf{q}^{x}_{0}$. *Then, we have*

$$\mathbf{q}^{\bar{x}}_{t+1/t} = \left[\left[\mathbf{q}^{\bar{x}}_{t+1/t}\right]_1 \ \left[\mathbf{q}^{\bar{x}}_{t+1/t}\right]_2 \ \cdots \ \left[\mathbf{q}^{\bar{x}}_{t+1/t}\right]_{N^{d+1}}\right]^{\mathsf{T}}, \tag{20}$$

$$\mathbf{q}^{\bar{x}}_{t/t} = \begin{cases} \mathbf{q}^{\bar{x}}_{t/t-1}, & y_t = \phi, \\ \left[\left[\mathbf{q}^{\bar{x}}_{t/t}\right]_1 \ \cdots \ \left[\mathbf{q}^{\bar{x}}_{t/t}\right]_{N^{d+1}}\right]^{\mathsf{T}}, & y_t = \delta_Q^{\lambda_t}, \end{cases} \tag{21}$$

where $\lambda_t \in [1:Q]$, $\left[\mathbf{q}^{\bar{x}}_{t+1/t}\right]_k = \sum_{j=1}^{N}\left[Lu_t\mathbf{q}^{w}\delta_N^{i_1}\left[\mathbf{q}^{\bar{x}}_{t/t}\right]_{\tilde{k}+j}\right]_{i_0}$, $\tilde{k} = \phi_N(i_1, i_2, \ldots, i_d, 0)$ *and* $\left[\mathbf{q}^{\bar{x}}_{t/t}\right]_k = \dfrac{\left[\tilde{H}_0\delta_N^{i_0}+\cdots+\tilde{H}_d\delta_N^{i_d}\right]_{\lambda_t}\left[\mathbf{q}^{\bar{x}}_{t/t-1}\right]_k}{\sum_{j_0,\ldots,j_d\in[1:N]}\left[\tilde{H}_0\delta_N^{j_0}+\cdots+\tilde{H}_d\delta_N^{j_d}\right]_{\lambda_t}\left[\mathbf{q}^{\bar{x}}_{t/t-1}\right]_l}$, $i_0, i_1, \ldots, i_d \in [1:N]$, $k = \phi_N(i_0, i_1, \ldots, i_d)$ *and* $l = \phi_N(j_0, j_1, \ldots, j_d)$,

$$\tilde{H} = H\mathbf{q}^{v}\left[\frac{\tau^0}{\tau^0+\tau^1+\cdots+\tau^d} \ \cdots \ \frac{\tau^d}{\tau^0+\tau^1+\cdots+\tau^d}\right] := [\tilde{H}_0 \ \tilde{H}_1 \ \cdots \ \tilde{H}_d].$$

Theorem 3.2

Consider BCN (16) with measurements (18). The optimal state estimation $\hat{x}_{t/t}$ *is given as follows:*

$$\hat{x}_{t/t} = \delta_N^{\hat{j}_t},$$

where $\hat{j}_t = \arg\min_{i\in[1:N]} \left(V_d \circ V_d\right)^{\mathsf{T}} \left[\mathbf{q}^{x}_{t/t}\right]_i$, $\left[\mathbf{q}^{x}_{t/t}\right]_i = \sum_{l=1}^{N^d} \left[\mathbf{q}^{\bar{x}}_{t/t}\right]_{(i-1)N^d+l}$ *and* $\mathbf{q}^{\bar{x}}_{t/t}$ *are obtained by Theorem 3.1.*

Estimate the state of BCN (16) from measurement (19)

The measurement (19) can be expressed as follows:

$$y(t) = Hv(t)C^{d(t)}\bar{x}(t), \ d(t) \in [0:d],$$

where $C^{d(t)}\bar{x}(t) = x(t - d(t))$, i.e.,

$$C^0 = [I_N \ 0 \ \cdots \ 0], \dots, C^d = [0 \ 0 \ \cdots \ I_N].$$

Here, $d(t)$ follows a discrete-time Markov chain taking values in a finite state-space $\{0, 1, \dots, d\}$. The transition probability matrix of $d(t)$ is $\Lambda = [\lambda_{ij}]$, where

$$\lambda_{ij} = \mathbb{P}\left(d(t) = j | d(t-1) = i\right), \ i, j \in [0:d].$$

Let $\mathrm{Col}_j(\Lambda)$ be the jth column of the transition probability matrix Λ. Then we set $\pi_i(t) = \mathbb{P}(d(t) = i)$ and denote $\pi(t) := [\pi_0(t) \ \pi_1(t) \ \cdots \ \pi_d(t)]^\mathsf{T}$.

$$
\begin{aligned}
\mathbb{P}(y(t) = Hv(t)x(t-i)) =& \mathbb{P}(d(t) = i) \\
=& \sum_{j=0}^{d} \mathbb{P}(d(t-1) = j)\mathbb{P}(d(t) = i | d(t-1) = j) \\
=& \sum_{j=0}^{d} \pi_j(t-1)\lambda_{ji} \\
=& [\mathrm{Col}_{i+1}(\Lambda)]^{\mathsf{T}} \pi(t-1), \ i \in [0:d].
\end{aligned}
$$

Here, we assume that $d(0)$ is a random variable following the probability distribution vector $\pi(0) = [\pi_0(0) \ \pi_1(0) \ \cdots \ \pi_d(0)]^{\mathsf{T}}$, i.e., $\mathbb{P}(d(0) = i) = \pi_i(0), \ i \in [0:d]$.

Theorem 3.3

Consider BCN (16) with measurements (19). Let $\mathbf{q}^{\bar{x}}_{0/-1}$ *be the initial probability distribution vector of the augmented state* $\bar{x}(t)$ *and* $\mathbf{q}^{x}_{0/-1} = \mathbf{q}^{x}_{0}$. *Assume that* $\pi(0) = [\pi_0(0)\ \pi_1(0)\ \cdots\ \pi_d(0)]^{\mathsf{T}}$. *Then, we have*

$$\mathbf{q}^{\bar{x}}_{t+1/t} = \left[\left[\mathbf{q}^{\bar{x}}_{t+1/t}\right]_1\ \left[\mathbf{q}^{\bar{x}}_{t+1/t}\right]_2\ \cdots\ \left[\mathbf{q}^{\bar{x}}_{t+1/t}\right]_{N^{d+1}}\right]^{\mathsf{T}}, \tag{22}$$

$$\mathbf{q}^{\bar{x}}_{t/t} = \left[\left[\mathbf{q}^{\bar{x}}_{t/t}\right]_1\ \left[\mathbf{q}^{\bar{x}}_{t/t}\right]_2\ \cdots\ \left[\mathbf{q}^{\bar{x}}_{t/t}\right]_{N^{d+1}}\right]^{\mathsf{T}}, \tag{23}$$

where $y_t = \delta_Q^{\lambda_t}$, $\lambda_t \in [1:Q]$, $\left[\mathbf{q}^{\bar{x}}_{t+1/t}\right]_k = \sum_{j=1}^{N}\left[L u_t \mathbf{q}^{w} \delta_N^{i_1}\left[\mathbf{q}^{\bar{x}}_{t/t}\right]_{\tilde{k}+j}\right]_{i_0}$, $\tilde{k} = \phi_N(i_1, i_2, \ldots, i_d, 0)$ *and* $\left[\mathbf{q}^{\bar{x}}_{t/t}\right]_k = \dfrac{\left[\tilde{H}_0(t)\delta_N^{i_0}+\cdots+\tilde{H}_d(t)\delta_N^{i_d}\right]_{\lambda_t}\left[\mathbf{q}^{\bar{x}}_{t/t-1}\right]_k}{\sum_{j_0,\ldots,j_d\in[1:N]}\left[\tilde{H}_0(t)\delta_N^{j_0}+\cdots+\tilde{H}_d(t)\delta_N^{j_d}\right]_{\lambda_t}\left[\mathbf{q}^{\bar{x}}_{t/t-1}\right]_l}$,
$i_0, i_1, \ldots, i_d \in [1:N]$, $k = \phi_N(i_0, i_1, \ldots, i_d)$, $l = \phi_N(j_0, j_1, \ldots, j_d)$,

$$\tilde{H}(t) = H\mathbf{q}^{v}\left[\pi_0(t)\ \pi_1(t)\ \cdots\ \pi_d(t)\right] := [\tilde{H}_0(t)\ \tilde{H}_1(t)\ \cdots\ \tilde{H}_d(t)],$$

$\pi_i(t) = [\mathrm{Col}_{i+1}(\Lambda)]^{\mathsf{T}}\pi(t-1)$, $i \in [0:d]$.

Application to state estimation of BCN (16) with the sampled measurement

The measurement received by the filter is considered as below:

$$y(t) = Hv(t)x(\theta_i),\ \theta_i \le t < \theta_{i+1}, \tag{24}$$

where $\theta_0 = 0,\ \theta_i,\ i \in \mathbb{Z}_+$ is uncertain and $\theta_{i+1} - \theta_i \in [1 : d+1]$. Suppose that $p_j,\ j \in [1 : d+1]$ is the activation frequency of the $\theta_{i+1} - \theta_i = j$, where $\sum_{j=1}^{d+1} p_j = 1$. Then Eq. (24) is rewritten as $y(t) = Hv(t)x(t - d(t)),\ \theta_i \le t < \theta_{i+1}$, where $d(t) := t - \theta_i$ is a random variable and it follows a discrete-time Markov chain taking values in a finite set $\{0, 1, \ldots, d\}$. Moreover, the transition probability matrix of $d(t)$ is

$$\Lambda = \begin{bmatrix} \frac{p_1}{\sum_{j=1}^{d+1} \frac{p_j}{j}} & \frac{\sum_{j=2}^{d+1} \frac{p_j}{j}}{\sum_{j=1}^{d+1} \frac{p_j}{j}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\frac{p_d}{d}}{\frac{p_d}{d} + \frac{p_{d+1}}{d+1}} & 0 & 0 & \cdots & \frac{\frac{p_{d+1}}{d+1}}{\frac{p_d}{d} + \frac{p_{d+1}}{d+1}} \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Here, we should note that $d(0) = 0$, i.e., $\pi(0) = [1\ 0\ \cdots\ 0]^{\mathsf{T}}$.

Conclusions

In **[S3]**, we investigate the optimal state estimation issue of BCNs with stochastic disturbances coming from measurements with random delay. The main contributions of this part are as follows:

1. In the previous existing literature on state estimation of BCNs, it was almost assumed that the measurements were received without delay. However, we found that in practice, measurements are always received with delay. For the first time, we focus on the measurements with random delay for state estimation problem of BCNs.
2. The sampled measurements are also considered for the first time. Notably, for the sampled measurements, we can transform them into the second type of measurements with random delay.

[S3] Liangjie Sun and Wai-Ki Ching, "State estimation of Boolean control networks under stochastic disturbances with random delay in measurements," *International Journal of Robust and Nonlinear Control*, vol. 33, no. 3, pp. 2447–2464, 2023.

Thanks