# Analysis of Discrete-Time Switched Linear Systems under Logical Dynamic Switching

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# WHAT IS SLS?

An SLS is a dynamic system that consists of a finite number of linear subsystems and a logical rule that regulates the activating subsystem during the operating time.

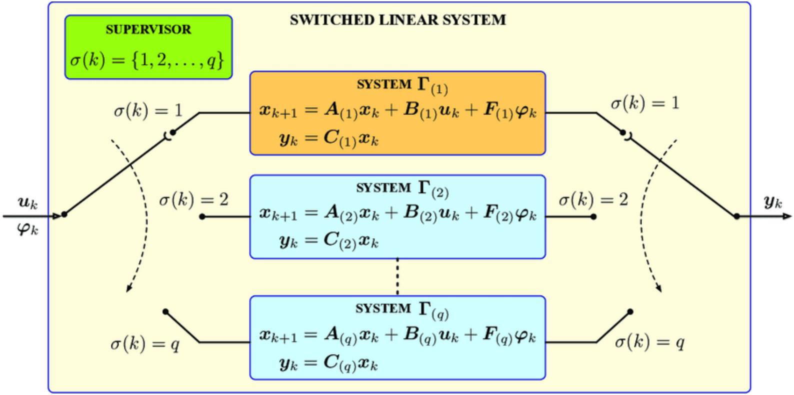


Figure 1: A switched linear system

# WHY STUDY SLS?

- Due to the flexibility of selecting the modes, SLS offers superior performance and can accomplish an enhanced range of tasks compared with each individual subsystem.
- In the majority of the existing literature investigating SLS, the logical rules that generate the switching signals were usually chosen freely or decided by the range of the physical state, i.e., the rules are given as piecewise constant maps from each switching time to the index set of the subsystems or state-feedback switching.

# WHAT PROBLEMS?

- Stability & Stabilization: [1, 2]
- Reachability & Controllability: [3, 4]
- Observability: [5, 6]


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# SLS under dynamical logic switching

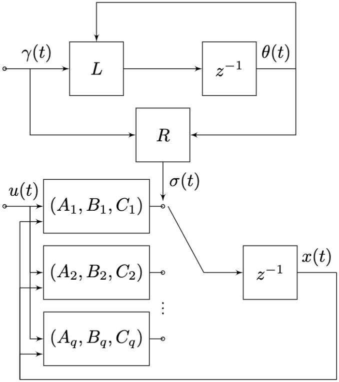


Figure 2: System diagram

# SLS under dynamical logic switching (Cont'd)

- SLS:

$$\begin{cases} x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \\ y(t) = C_{\sigma(t)}x(t), \end{cases} \tag{1}$$

- LCN:

$$\begin{cases} \vec{\theta}(t+1) = L \ltimes \vec{\gamma}(t) \ltimes \vec{\theta}(t), \\ \vec{\sigma}(t) = R \ltimes \vec{\gamma}(t) \ltimes \vec{\theta}(t), \end{cases} \tag{2}$$

# STP-Based Mergence

Letting $z(t) := \vec{\theta}(t) \ltimes x(t) \in \mathbb{R}^{Nn}$, the STP-based mergence of the systems is derived as follows:

$$\begin{aligned} z(t+1) &= \vec{\theta}(t+1) \ltimes x(t+1) \\ &= L\vec{\gamma}(t)\vec{\theta}(t)[\mathbf{A}\vec{\sigma}(t)x(t) + \mathbf{B}\vec{\sigma}(t)u(t)] \\ &= L\vec{\gamma}(t)\vec{\theta}(t)\mathbf{A}R\vec{\gamma}(t)\vec{\theta}(t)x(t) \\ &\quad + L\vec{\gamma}(t)\vec{\theta}(t)\mathbf{B}R\vec{\gamma}(t)\vec{\theta}(t)u(t) \\ &= L[I_{MN} \otimes (\mathbf{A}R)]\Phi_{MN}\vec{\gamma}(t)\vec{\theta}(t)x(t) \\ &\quad + L[I_{MN} \otimes (\mathbf{B}R)]\Phi_{MN}\vec{\gamma}(t)\vec{\theta}(t)u(t) \\ &= \mathbf{G}\vec{\gamma}(t)z(t) + \mathbf{H}\vec{\gamma}(t)\vec{\theta}(t)u(t), \end{aligned} \tag{3}$$

where $\mathbf{G} := L[I_{MN} \otimes (\mathbf{A}R)]\Phi_{MN} \in \mathcal{M}_{nN\times nMN}$,
$\mathbf{H} := L[I_{MN} \otimes (\mathbf{B}R)]\Phi_{MN} \in \mathcal{M}_{nN\times mMN}$.

# STP-Based Mergence (Cont'd)

Since $\vec{\gamma}(t) \in \Delta_M$, based on the properties of the STP, the merged system can be rewritten as:

$$\begin{aligned} z(t+1) &= [G_1\ G_2\ \cdots\ G_M]\vec{\gamma}(t)z(t) \\ &\quad +[H_1\ H_2\ \cdots\ H_M]\vec{\gamma}(t)\vec{\theta}(t)u(t), \end{aligned} \tag{4}$$

where $G_i = \begin{bmatrix} G^i_{1,1} & G^i_{1,2} & \cdots & G^i_{1,N} \\ G^i_{2,1} & G^i_{2,2} & \cdots & G^i_{2,N} \\ \vdots & \vdots & & \vdots \\ G^i_{N,1} & G^i_{N,2} & \cdots & G^i_{N,N} \end{bmatrix}$, and $H_i = \begin{bmatrix} H^i_{1,1} & H^i_{1,2} & \cdots & H^i_{1,N} \\ H^i_{2,1} & H^i_{2,2} & \cdots & H^i_{2,N} \\ \vdots & \vdots & & \vdots \\ H^i_{N,1} & H^i_{N,2} & \cdots & H^i_{N,N} \end{bmatrix}$,
each submatrix $G^i_{\alpha,\beta} \in \mathcal{M}_{n\times n}$ and $H^i_{\alpha,\beta} \in \mathcal{M}_{n\times m}$, $\alpha, \beta \in [1, N]$.
We call the above matrices the block form of $G_i$ and $H_i$, $i = 1, \cdots, M$.

# Reachability Problem

**Definition 1**

Consider the switched linear system (1) in which the switching signal is generated by the logical dynamical system (2).

1. A state $x \in \mathbb{R}^n$ is reachable, if there exists a positive integer $T < \infty$, a logical input sequence $(\gamma_0, \gamma_1, \cdots, \gamma_{T-1})$, and an input sequence $(u_0, u_1, \cdots, u_{T-1})$, such that for $\forall \vec{\theta}_0 \in \Delta_N$,

$$x_0 = \mathbf{0}, \quad x_T = x.$$

2. The system is reachable if all states in $\mathbb{R}^n$ are reachable.

**Definition 2**

Consider the merged system (4). Define the reachable set of state $x = \mathbf{0}$ with $T$-length logical input sequence $(\gamma_0, \gamma_1, \cdots, \gamma_{T-1})$ and initial logical state $\alpha \in [1, N]$ as

$$\begin{array}{c} \mathcal{R}_T^\alpha(\gamma_0, \gamma_1 \cdots, \gamma_{T-1}) := \\ \mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}(G_{\gamma_{T-1}} G_{\gamma_{T-2}} \cdots G_{\gamma_1} H_{\gamma_0} \delta_N^\alpha) \right] \cup \\ \mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}(G_{\gamma_{T-1}} G_{\gamma_{T-2}} \cdots G_{\gamma_2} H_{\gamma_1} L_{\gamma_0} \delta_N^\alpha) \right] \cup \\ \vdots \\ \cup \mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}(G_{\gamma_{T-1}} H_{\gamma_{T-2}} L_{\gamma_{T-3}} \cdots L_{\gamma_1} L_{\gamma_0} \delta_N^\alpha) \right] \\ \cup \mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}(H_{\gamma_{T-1}} L_{\gamma_{T-2}} \cdots L_{\gamma_0} \delta_N^\alpha) \right]. \end{array} \tag{5}$$

**Theorem 1**

The switched linear system (1) is reachable under the logically generated switching signal, if and only if there exists a logical input sequence $(\gamma_0, \gamma_1, \cdots, \gamma_{T-1})$, such that the reachable set of the merged system satisfies

$$\bigcap_{\alpha=1}^{N} \mathcal{R}_T^{\alpha}(\gamma_0, \gamma_1 \cdots, \gamma_{T-1}) = \mathbb{R}^n. \tag{6}$$

# SLS under dynamical logic switching

**Algorithm 1** Computation of feasible logical input sequences

**Input:** G, $N$, $\mathcal{A}$
%structucre matrix, number of states, control attractor set
%(computed by Lemma 3)
**Output:** $(\gamma_0^j, \cdots, \gamma_{k-2}^j, \gamma_{k-1}^j)$

**for** $k = 1 : N$ **do**
  $\mathcal{G}_k := \left(\sum_{i=1}^{M} G_i\right)^k$ %use symbolic operation here
  **for** $\alpha \in \mathcal{A}$ **do**
    **if** $\bigcap_{\alpha \in \mathcal{A}} \mathcal{R}_T^{\alpha}(\gamma_0^j, \cdots, \gamma_{k-2}^j, \gamma_{k-1}^j) = \mathbb{R}^n$. **then**
      Print: "A feasible input sequence is $(\gamma_0^j, \cdots, \gamma_{k-2}^j, \gamma_{k-1}^j)$."
      **return**
    **end if**
  **end for**
  **if** $k = N$ **then**
    Print: "There is no feasible input sequence, the system is not globally reachable."
    **return**
  **end if**
**end for**
**return**

# Controllability Problem

**Definition 3**

Consider the switched linear system (1) in which the switching signal is generated by the logical dynamical system (2).

1. A state $x \in \mathbb{R}^n$ is controllable, if there exist a positive integer $T < \infty$, a logical input sequence $(\gamma_0, \gamma_1, \cdots, \gamma_{T-1})$, and an input sequence $(u_0, u_1, \cdots, u_{T-1})$, such that for $\forall \vec{\theta}_0 \in \Delta_N$,

$$x_0 = x, \quad x_T = \mathbf{0}.$$

2. The system is controllable if all the states in $\mathbb{R}^n$ are controllable.

## Theorem 2

The switched linear system (1) is controllable under the logically generated switching signal, if and only if there exists a logical input sequence $(\gamma_0, \gamma_1, \cdots, \gamma_{T-1})$, $T < \infty$, such that starting from any logical state $\alpha \in [1, N]$, the merged system satisfies

$$\mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}( G_{\gamma_{T-1}} G_{\gamma_{T-2}} \cdots G_{\gamma_0} \delta_N^{\alpha} ) \right] \subset \mathcal{R}_T^{\alpha}(\gamma_0, \gamma_1 \cdots, \gamma_{T-1}). \tag{7}$$

# Observability & Reconstructibility Problems

**Definition 4**

Consider the switched linear system (1) where the switching signal is generated by the logical dynamical system (2). The system is observable (reconstructible), if there exists a positive integer $T < \infty$ and a logical input sequence $(\gamma_0, \gamma_1, \cdots, \gamma_{T-1})$, such that the input sequence $(u_0, u_1, \cdots, u_{T-1})$ and output trajectory $(y_0, y_1, \cdots, y_T)$ can uniquely determine the initial state $x_0$ $(x_T)$, regardless of the value of the initial logical state $\theta_0$.

Now we build a dual SLS as

$$\tilde{x}(t+1) = \tilde{\mathbf{A}} \ltimes \vec{\sigma}(t) \ltimes \tilde{x}(t) + \tilde{\mathbf{C}} \ltimes \vec{\sigma}(t) \ltimes \tilde{u}(t), \qquad (8)$$

where $\tilde{x} \in \mathbb{R}^n$, $\tilde{u} \in \mathbb{R}^p$, $\tilde{\mathbf{A}} = [\tilde{A}_1\ \tilde{A}_2\ \cdots\ \tilde{A}_q] := [A_1^{\mathrm{T}}\ A_2^{\mathrm{T}}\ \cdots\ A_q^{\mathrm{T}}]$, $\tilde{\mathbf{C}} = [\tilde{C}_1\ \tilde{C}_2\ \cdots\ \tilde{C}_q] := [C_1^{\mathrm{T}}\ C_2^{\mathrm{T}}\ \cdots\ C_q^{\mathrm{T}}]$.
Then we can also build a merged system:

$$\begin{aligned}
\tilde{z}(t+1) &= \vec{\theta}(t+1) \ltimes \tilde{x}(t+1) \\
&= L\vec{\gamma}(t)\vec{\theta}(t)[\tilde{\mathbf{A}}\vec{\sigma}(t)\tilde{x}(t) + \tilde{\mathbf{C}}\vec{\sigma}(t)\tilde{u}(t)] \\
&= L\vec{\gamma}(t)\vec{\theta}(t)\tilde{\mathbf{A}}R\vec{\gamma}(t)\vec{\theta}(t)\tilde{x}(t) \\
&\quad + L\vec{\gamma}(t)\vec{\theta}(t)\tilde{\mathbf{C}}R\vec{\gamma}(t)\vec{\theta}(t)\tilde{u}(t) \\
&= L[I_{MN} \otimes (\tilde{\mathbf{A}}R)]\Phi_{MN}\vec{\gamma}(t)\vec{\theta}(t)\tilde{x}(t) \\
&\quad + L[I_{MN} \otimes (\tilde{\mathbf{C}}R)]\Phi_{MN}\vec{\gamma}(t)\vec{\theta}(t)\tilde{u}(t) \\
&= \tilde{\mathbf{G}}\vec{\gamma}(t)\tilde{z}(t) + \tilde{\mathbf{H}}\vec{\gamma}(t)\vec{\theta}(t)\tilde{u}(t),
\end{aligned} \qquad (9)$$

where $\tilde{\mathbf{G}} := L[I_{MN} \otimes (\tilde{\mathbf{A}}R)]\Phi_{MN} \in \mathcal{M}_{nN\times nMN}$,
$\tilde{\mathbf{H}} := L[I_{MN} \otimes (\tilde{\mathbf{C}}R)]\Phi_{MN} \in \mathcal{M}_{nN\times mMN}$.

## Definition 5

Define the reachable set of state $\tilde{x}$ with $T$-length logical input sequence $(\gamma_0, \gamma_1, \cdots, \gamma_{T-1})$ and initial logical state $\alpha \in [1, N]$ as

$$\begin{aligned}
&\tilde{\mathcal{R}}_T^{\alpha}(\gamma_0, \gamma_1 \cdots, \gamma_{T-1}) := \\
&\mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}(\tilde{G}_{\gamma_0} \tilde{G}_{\gamma_1} \cdots \tilde{G}_{\gamma_{T-2}} \tilde{H}_{\gamma_{T-1}} L_{\gamma_{T-2}} \cdots L_{\gamma_0} \delta_N^{\alpha}) \right] \cup \\
&\mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}(\tilde{G}_{\gamma_0} \tilde{G}_{\gamma_1} \cdots \tilde{G}_{\gamma_{T-3}} \tilde{H}_{\gamma_{T-2}} L_{\gamma_{T-3}} \cdots L_{\gamma_0} \delta_N^{\alpha}) \right] \cup \\
&\vdots \\
&\cup \mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}(\tilde{G}_{\gamma_0} \tilde{H}_{\gamma_1} L_{\gamma_0} \delta_N^{\alpha}) \right] \cup \mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}(\tilde{H}_{\gamma_0} \delta_N^{\alpha}) \right].
\end{aligned} \tag{10}$$

**Theorem 3**

The switched linear system (1) is observable under the logically generated switching signal, if and only if there exists a logical input sequence $(\gamma_0, \gamma_1, \cdots, \gamma_{T-1})$, such that the reachable set of the merged system satisfies

$$\bigcap_{\alpha=1}^{N} \tilde{\mathcal{R}}_T^{\alpha}(\gamma_0, \gamma_1 \cdots, \gamma_{T-1}) = \mathbb{R}^n. \tag{11}$$

**Theorem 4**

The switched linear system (1) is reconstructible under the logically generated switching signal, if and only if there exists a logical input sequence $(\gamma_0, \gamma_1, \cdots, \gamma_{T-1})$, $T < \infty$, such that staring from any logical state $\alpha \in [1, N]$, the merged system satisfies

$$\mathbf{1}_N^{\mathrm{T}} \left[ \mathrm{Im}(\tilde{G}_{\gamma_0} \tilde{G}_{\gamma_1} \cdots \tilde{G}_{\gamma_{T-1}}) \right] \subset \tilde{\mathcal{R}}_T^{\alpha}(\gamma_0, \gamma_1 \cdots, \gamma_{T-1}). \tag{12}$$

# System Realization Problem

We consider two kinds of logical regulating methods:

- *Case 1:* Generating the switching signal sequences that guarantee the fixed operating times (FOTs) for the subsystems.
- *Case 2:* Generating the switching signals aligned to a finite reference sequence.

# $\ell$-step Input-State Set Reachability

Consider the logical dynamical system. Denote the input set by $\mathcal{U} := \{1, 2, \cdots, M\}$ and the state set by $\mathcal{X} := \{1, 2, \cdots, N\}$, then $\Omega \in 2^{\mathcal{U}\times\mathcal{X}} \setminus \{\emptyset\}$ is an input-state subset ($\mathcal{U} \times \mathcal{X}$ is isomorphic to $\Delta_{MN}$ under logical operations). Now we denote $V(\Omega) \in \mathcal{B}_{MN\times 1}$ the index vector of $\Omega$, which is defined as

$$[V(\Omega)]_i := \begin{cases} 1, & i \in \Omega; \\ 0, & i \notin \Omega. \end{cases}$$

One can see that $V(\Omega) = \sum_{(\gamma,\theta)\in\Omega} \vec{\gamma}\vec{\theta}$.

For a class of initial state subsets

$$\Omega^0 := \{\Omega_1^0, \Omega_2^0, \cdots \Omega_\alpha^0\}$$

and a class of terminal state subsets

$$\Omega^d := \{\Omega_1^d, \Omega_2^d, \cdots \Omega_\beta^d\},$$

where $\alpha, \beta \in \mathbb{Z}_+$ are the numbers of the subsets, define their index matrices as

$$\begin{aligned} P_{\Omega^0} &:= \begin{bmatrix} V(\Omega_1^0) & V(\Omega_2^0) & \cdots & V(\Omega_\alpha^0) \end{bmatrix} \in \mathcal{B}_{MN\times\alpha}, \\ P_{\Omega^d} &:= \begin{bmatrix} V(\Omega_1^d) & V(\Omega_2^d) & \cdots & V(\Omega_\beta^d) \end{bmatrix} \in \mathcal{B}_{MN\times\beta}. \end{aligned}$$

Now we are ready to give the concept of input-state set reachability of logical dynamical systems.

**Definition 5**

Consider the logical dynamical system (2) with a class of initial input-state subsets $\Omega^0 = \{\Omega^0_1, \Omega^0_2, \cdots \Omega^0_\alpha\}$ and a class of terminal input-state subsets $\Omega^d = \{\Omega^d_1, \Omega^d_2, \cdots \Omega^d_\beta\}$.

1. The system is $\ell$-step input-state reachable from $(\gamma^0, \theta^0)$ to $(\gamma^d, \theta^d)$ if there exists at least a logical input sequence $(\gamma(0), \gamma(1), \cdots, \gamma(\ell))$, where $\gamma(0) = \gamma^0$ and $\gamma(\ell) = \gamma^d$, such that $(\gamma^0, \theta^0)$ can be steered to $(\gamma^d, \theta^d)$.
2. The system is $\ell$-step input-state set reachable from $\Omega^0_j$ to $\Omega^d_i$ if for some $(\gamma^0, \theta^0) \in \Omega^0_j$ and some $(\gamma^d, \theta^d) \in \Omega^d_i$ there exists at least a logical input sequence $(\gamma(0), \gamma(1), \cdots, \gamma(\ell))$, where $\gamma(0) = \gamma^0$ and $\gamma(\ell) = \gamma^d$, such that the system is reachable from $(\gamma^0, \theta^0)$ to $(\gamma^d, \theta^d)$.
3. The system is $\ell$-step input-state set reachable at $\Omega^0_j$ if the system is set reachable from $\Omega^0_j$ to $\forall \Omega^d_i \in \Omega^d$.
4. $\Omega^d_i$ is globally $\ell$-step input-state set reachable if the system is set reachable from $\forall \Omega^0_j \in \Omega^0$ to $\Omega^d_i$.
5. The system is $\ell$-step input-state set reachable from $\Omega^0$ to $\Omega^d$ if for $\forall \Omega^0_j \in \Omega^0$ and $\forall \Omega^d_i \in \Omega^d$, the system is $\ell$-step input-state set reachable from $\Omega^0_j$ to $\Omega^d_i$.

Define the input-state matrix of the logical dynamical system (2) as

$$\mathbf{L} := \mathbf{1}_M L = \underbrace{[L^{\mathrm{T}}\ L^{\mathrm{T}}\ \cdots\ L^{\mathrm{T}}]}_{M}^{\mathrm{T}}. \tag{13}$$

Given $\Omega^0$ and $\Omega^d$, we have the $\ell$-step input-state set reachability matrix of the system (2) as

$$\mathcal{C}_\ell := (P_\Omega^d)^{\mathrm{T}} \times_{\mathcal{B}} \mathbf{L}^{(\ell)} \times_{\mathcal{B}} P_\Omega^0, \tag{14}$$

the $\ell$-step input-state set reachability can be verified by the following conditions.

## Proposition 1

Consider the logical dynamical system (2) with a group of initial input-state subsets $\{\Omega_1^0, \Omega_2^0, \cdots \Omega_\alpha^0\}$ and terminal input-state subsets $\{\Omega_1^d, \Omega_2^d, \cdots \Omega_\beta^d\}$.

1. The system is $\ell$-step input-state set reachable from $\Omega_j^0$ to $\Omega_i^d$, if and only if $[\mathcal{C}_\ell]_{i,j} = 1$.
2. The system is $\ell$-step input-state set reachable at $\Omega_j^0$, if and only if $\mathrm{Col}_j(\mathcal{C}_\ell) = \mathbf{1}_\beta$.
3. $\Omega_i^d$ is global $\ell$-step input-state set reachable, if and only if $\mathrm{Row}_i(\mathcal{C}_\ell) = \mathbf{1}_\alpha^{\mathrm{T}}$.
4. The system is $\ell$-step input-state set reachable, if and only if $\mathcal{C}_\ell = \mathbf{1}_{\beta\times\alpha}$.

# Fixed Operating Time Switching

**Theorem 5**

Consider the system (1) in which the switching signal is generated by the system (2). Denote by $d_1, d_2, \cdots, d_q \in \mathbb{Z}_+ \cup \{\infty\}$ the FOTs of the subsystems $\Sigma_1, \Sigma_2, \cdots, \Sigma_q$. The switching signal sequences, which ensure the FOTs, can be generated by the system (2) under some logical inputs, no matter what the initial logical state is, if and only if

1. For the subsystem $\Sigma_i$ whose FOT $d_i = 1$, there exists at least one logical input $\gamma \in [1, M]$, such that the switching signal $\sigma = i$ can be transferred to other values.
2. For the subsystem $\Sigma_i$ whose FOT satisfies $1 < d_i < \infty$, there exists at least one logical input $\gamma \in [1, M]$, such that the switching signal $\sigma = i$ can be transferred to other values; and exists at least one logical input $\gamma \in [1, M]$, such that the switching signal $\sigma = i$ can keep still.
3. For the subsystem $\Sigma_i$ whose FOT satisfies $d_i = \infty$, there exists at least one logical input $\gamma \in [1, M]$, such that the switching signal $\sigma = i$ can keep still.

Let $\mathcal{I} := \{i \mid d_i = 1\}$, $\mathcal{J} := \{i \mid 1 < d_i < \infty\}$, and $\mathcal{K} := \{i \mid d_i = \infty\}$. Define a series of input-state subsets as $\mathcal{O}_i := \{\delta_{MN}^{\alpha} \mid R\delta_{MN}^{\alpha} = \delta_q^i\}$, and define the singleton version of $\mathcal{O}_i$ as $\tilde{\mathcal{O}}_i := \{\{\delta_{MN}^{\alpha}\} \mid R\delta_{MN}^{\alpha} = \delta_q^i\}$.

**Theorem 6**

The switching signal sequence which guarantees the FOTs of the switched linear system (1) can be generated regardless of the initial logical state of the logical dynamical system (2), if and only if

1. $\forall i \in \mathcal{I}$, the system is 1-step input-state set reachable from every singleton in $\tilde{\mathcal{O}}_i$ to $\Delta_{MN} \setminus \mathcal{O}_i$, that is,

$$P_{\{\Delta_{MN}\setminus\mathcal{O}_i\}}^{\mathrm{T}} \times_{\mathcal{B}} \mathbf{L} \times_{\mathcal{B}} P_{\tilde{\mathcal{O}}_i} = \mathbf{1}_{|\mathcal{O}_i|}^{\mathrm{T}}. \tag{15}$$

2. $\forall i \in \mathcal{J}$, the system is 1-step input-state set reachable from every singleton in $\tilde{\mathcal{O}}_i$ to both $\Delta_{MN} \setminus \mathcal{O}_i$ and $\mathcal{O}_i$, that is,

$$\begin{cases} P_{\{\Delta_{MN}\setminus\mathcal{O}_i\}}^{\mathrm{T}} \times_{\mathcal{B}} \mathbf{L} \times_{\mathcal{B}} P_{\tilde{\mathcal{O}}_i} = \mathbf{1}_{|\mathcal{O}_i|}^{\mathrm{T}}, \\ P_{\{\mathcal{O}_i\}}^{\mathrm{T}} \times_{\mathcal{B}} \mathbf{L} \times_{\mathcal{B}} P_{\tilde{\mathcal{O}}_i} = \mathbf{1}_{|\mathcal{O}_i|}^{\mathrm{T}}. \end{cases} \tag{16}$$

3. $\forall i \in \mathcal{K}$, the system is 1-step input-state set reachable from singleton in $\tilde{\mathcal{O}}_i$ to $\mathcal{O}_i$, that is,

$$P_{\{\mathcal{O}_i\}}^{\mathrm{T}} \times_{\mathcal{B}} \mathbf{L} \times_{\mathcal{B}} P_{\tilde{\mathcal{O}}_i} = \mathbf{1}_{|\mathcal{O}_i|}^{\mathrm{T}}. \tag{17}$$

where $\mathbf{L} = \mathbf{1}_M L = [\underbrace{L^{\mathrm{T}}, L^{\mathrm{T}}, \cdots, L^{\mathrm{T}}}_{M}]^{\mathrm{T}}$ is the input-state matrix of the logical dynamical system (2).

# Finite Reference Signal Switching

**Definition 7**

Consider the logical dynamical system (2) with an initial state $\theta_0$. The reference signal sequence $(\sigma_0, \sigma_1, \cdots, \sigma_\tau)$ is called trackable if there exists a logical input sequence $\Gamma := (\gamma_0, \gamma_1, \cdots, \gamma_\tau)$ such that

$$\sigma(t, \theta_0, \Gamma) = \sigma_t, \quad t = 0, 1, \cdots, \tau.$$

For a given reference signal sequence $(\sigma_0, \sigma_1, \cdots, \sigma_\tau)$, we define a series of input-state subsets as

$$\mathcal{O}_{\sigma_t} := \{\delta_{MN}^i \mid R\delta_{MN}^i = \sigma_t\}, \; t = 0, 1, \cdots, \tau.$$

**Theorem 7**

Consider the logical dynamical system (2) with an initial state $\theta_0$. The reference signal sequence $(\sigma_0, \sigma_1, \cdots, \sigma_\tau)$ is trackable, if and only if

$$P^{\mathrm{T}}_{\{\mathcal{O}_{\sigma_t}\}}\mathbf{L}\vec{\vartheta}(t-1) > 0, \quad t = 1, 2, \cdots, \tau, \tag{18}$$

where
$$\begin{cases} \vec{\vartheta}(0) = \left(\mathbf{1}_M \vec{\theta}(0)\right) \wedge P_{\{\mathcal{O}_{\sigma_0}\}} \\ \vec{\vartheta}(t) = \left(\mathbf{L} \times_{\mathcal{B}} \vec{\vartheta}(t-1)\right) \wedge P_{\{\mathcal{O}_{\sigma_t}\}} \end{cases}.$$

# Illustrative Example

Consider the SLS

$$\begin{cases} x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \\ y(t) = C_{\sigma(t)}x(t), \end{cases} \tag{19}$$

whose switching signals $\sigma(t) \in \{1, 2\}$ are generated by the following logical dynamical system

$$\begin{cases} \vec{\theta}(t+1) = L \ltimes \vec{\gamma}(t) \ltimes \vec{\theta}(t), \\ \vec{\sigma}(t) = R \ltimes \vec{\gamma}(t) \ltimes \vec{\theta}(t), \end{cases} \tag{20}$$

$$A_1 = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -2 & 2 & 1 \\ 0 & -2 & 0 \\ 1 & -4 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix},$$

$$L = \delta_4[1, 1, 2, 4, 4, 4, 3, 3],$$

$$R = \delta_2[2, 2, 1, 1, 1, 2, 2, 1].$$

With the equation (3), one has $\mathbf{G} = \left[G_1, G_2\right]$.

$$G_1 = \begin{bmatrix} -2 & 2 & 1 & -2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 3 \end{bmatrix} := \begin{bmatrix} G^1_{1,1} & G^1_{1,2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & G^1_{2,3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & G^1_{4,4} \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -4 & 0 & 1 & -4 & 3 \\ 1 & 2 & -1 & -2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 3 & 1 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & G^2_{3,3} & G^2_{3,4} \\ G^2_{4,1} & G^2_{4,2} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

We first check the control attractors, as well as their attractor basins, of the logical dynamical system (20). We can find that all the logical states are control attractors and the attract basin of $\delta_4^4$ is the whole state space $\Delta_4$. Hence it is only needed to check if the equations (6), (7), (11), and (12) hold for $\alpha = 4$.
Using Algorithm 1, we have

$$\mathcal{G}_3 = \begin{bmatrix} * & G^1_{1,2} G^1_{2,3} G^2_{3,4} \\ * & G^1_{2,3} G^2_{3,3} G^2_{3,4} + G^1_{2,3} G^2_{3,4} G^1_{4,4} \\ * & G^2_{3,3} G^2_{3,3} G^2_{3,4} + G^2_{3,3} G^2_{3,4} G^1_{4,4} + G^2_{3,4} G^1_{4,4} G^1_{4,4} \\ * & G^2_{4,2} G^1_{2,3} G^2_{3,4} + G^1_{4,4} G^1_{4,4} G^1_{4,4} \end{bmatrix},$$

where the first three columns of blocks are omitted because only the 4-th column of the block matters.

It can be found that the equations (6) and (7) hold for logical input sequences

$$(2,2,2),\ (2,2,1),\ (2,1,2),\ (2,1,1),\ (1,2,2).$$

Thus, the SLS is reachable and controllable.
Using a similar computation process, one can also conclude that the system is observable and reconstructible: with $\alpha = 4$, the equations (11), and (12) hold for all the 3-length logical input sequences except $(2,2,2)$.

# Thanks for your attention!