Logical Matrix Equations and Their Applications

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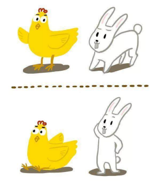
Outline

Section I: Background

The Chickens-and-Rabbits Problem



有雉兔同笼，
上有三十五头，
下有九十四足，
问雉兔各几何？
——《孙子算经》

There are several chickens and rabbits in the same cage. If there are a total of 35 heads and 94 legs in the cage, then find out the number of chickens and rabbits, respectively.

Assume the number of chickens is x and the number of rabbits is y.

$$\begin{cases} x+y = 35 \\ 2x+4y = 94 \end{cases}$$

$$\Downarrow$$

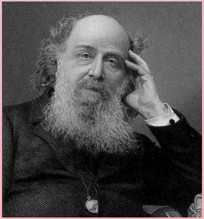
$$A\mathbf{x} = \mathbf{b},$$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 35 \\ 94 \end{bmatrix}$$

Gaussian elimination $\Rightarrow x = 23,\ y = 12$

Matrix Equations

- The Chinese text The Nine Chapters on the Mathematical Art is the first example of the use of array methods to solve simultaneous equations.
- In 1848, J.J. Sylvester introduced the term “matrix”.
- Lyapunov equation
  $AX + XA^\top + Q = 0$
- Sylvester equation
  $AX + XB = C$
- Algebraic Riccati equation
  $A^\top P + PA - PBR^{-1}B^\top P + Q = 0$



Subsection I.i: Semi-Tensor Product

Semi-tensor product of matrices

Definition 1

Semi-tensor product of matrices $M \in \mathbb{R}_{m\times n}$ and $N \in \mathbb{R}_{p\times q}$ is

$$M \ltimes N = (M \otimes I_{t/n})(N \otimes I_{t/p}) \in \mathbb{R}_{mt/n\times qt/p}, \tag{1}$$

where t is the least common multiple of n and p, and $\otimes$ is the Kronecker product.

D. Cheng, H. Qi, Z. Li, Analysis and Control of Boolean networks: A Semi-tensor Product Approach, Springer, 2011.

Applications on STP

- Boolean Network.
  Fixed points and cycles, controllability and observability, etc.
- Game Theory.
  Potential equation, etc.
- Shift Register.
  Hu-Gong's open problem, etc.
- IC Engine, Fuzzy System, Automata, Compressive Sensing, etc.

Y.Guo, Y.Wu, C.Li, W.Gui. IEEE T. Auto. Contr., 2024.
S.Zhu, J.Lu, D.Ho, J.Cao. IEEE T. Auto. Contr., 2024.
Y.Wu, S.Le, K.Zhang, X.Sun. IEEE T. Auto. Contr., 2022.
L.Li, L.Liu, H.Peng, Y.Yang, S.Cheng. IEEE Internet Things J., 2019.
J.Zhong, D.Lin. IEEE T. Inform. Theory, 2018.
X.Liu, J.Zhu. Automatica, 2016.
D.Cheng. Automatica, 2014.
D.Cheng, H.Qi. IEEE T. Auto. Contr., 2010.
D.Cheng, H.Qi. Automatica, 2009.

Subsection I.ii: Preliminaries

Algebraic Representation

Notations

$\mathcal{B} := \{0,1\}$, $\Delta_n := \{\delta_n^i | i = 1,2,\ldots,n\}$, where $\delta_n^i = \mathrm{Col}_i(I_n)$.
$\mathcal{L}^{m\times n}$: Set of all the logical matrices in dimensions $m \times n$.

Lemma 1

Let $f(x_1,x_2,\ldots,x_n) : \mathcal{B}^n \to \mathcal{B}$ be a Boolean function. Then there exists a unique matrix $M_f \in \mathcal{L}^{2\times 2^n}$, called the structure matrix of $f(\cdot)$, such that $\delta_2^{2-f(x_1,x_2,\ldots,x_n)} = M_f \ltimes_{i=1}^n x_i$, with $x_i = \delta_2^{2-x_i} \in \Delta_2$.

D. Cheng, H. Qi, Z. Li. Analysis and Control of BNs: A Semi-tensor Product Approach, Springer, 2011.

Boolean control network:

$$\begin{cases} x_i(t+1) = f_i(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)), \\ y_j(t) = h_j(x_1(t), \ldots, x_n(t)), \\ i \in [1, n], j \in [1, p]. \end{cases} \quad (2)$$

$f_i(\cdot)$ and $h_j(\cdot)$, $x_i(\cdot)$, $i \in [i, n]$, $j \in [1, p]$ are Boolean functions and Boolean variables, respectively.

- Using **Lemma 1**, the logical from of Boolean control network (2) can be expressed as following:

$$\begin{cases} x(t+1) = \bar{F}_1 * \bar{F}_2 * \cdots * \bar{F}_n x(t) = Lu(t)x(t), \\ y(t) = H_1 * H_2 * \cdots * H_p x(t) = Hx(t), \end{cases} \quad (3)$$

  with state $x(t) \in \Delta_{2^n}$, input $u(t) \in \Delta_{2^m}$ and output $y(t) \in \Delta_{2^p}$.

- $L := [F_1, F_2, \ldots, F_{2^m}]$.
- Denote $\mathcal{B}(L, H)$ BCN (3).
- Denote $\mathcal{L}(L, H)$ logical control network(LCN) (3).

Section II: Logical Matrix Equation

Example 1-Matrix Factorization

Logical matrix $A \in \mathcal{L}^{m\times n}$ is said to be factorizable, if there exist a positive integer p and two logical matrices $B \in \mathcal{L}^{m\times p}$ and $C \in \mathcal{L}^{p\times n}$, such that $A = BC$.

BN $\mathcal{B}(L, H) \Rightarrow L = L_1L_2 \Rightarrow \hat{L} = L_2L_1 \Rightarrow$

$$z(t+1) = \hat{L}z(t). \qquad (4)$$

$\mathcal{B}(L, H)$ and the logical system (4) have the same topological structure (including all the fixed points and cycles).

H. Li, Y, Wang. IEEE T. Auto. Contr., 2015.

Example 2-Coloring Problem

$\mathcal{G} = \{V, E\}$ is a undirected graph without loops.
The $k-$coloring problem: design a mapping $\mathrm{c}(\cdot) : V = \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ satisfying $\mathrm{c}(i) \neq \mathrm{c}(j)$ if $(i, j) \in E$.

$\mathcal{G}$ is $k-$colorable if and only if there exists $\mathrm{c}(\cdot)$ such that the following system of Boolean equations holds,

$$\begin{cases} \mathrm{c}(i)\bar{\nabla}\mathrm{c}(j) = 1, \\ (i, j) \in E, \end{cases} \Leftrightarrow \bigwedge_{(i,j)\in E} \mathrm{c}(i)\bar{\nabla}\mathrm{c}(j) = 1 \tag{5}$$

where $\bar{\nabla}$ is the XOR of $k-$valued logical operation.

Equations set (5) can be transformed into

$$Mx = \delta_2^1, \tag{6}$$

where $x = \ltimes_{i=1}^{n} \delta_k^{\mathrm{c}(i)}$.

Y. Wang, C. Zhang, Z. Liu. Automatica, 2012.

Example 3-Implicit Function Theorem

Consider a Boolean equation

$$L_I xy = \delta_2^1, \tag{7}$$

where $L_I \in \mathcal{L}^{2\times 2^n}$ and $x \in \Delta_{2^{n-r}}$, $y \in \Delta_{2^r}$, $1 \le r < n$.
Aim: Equivalently convert (7) into

$$y = M_I x, \tag{8}$$

where $M_I \in \mathcal{L}^{2^r \times 2^{n-r}}$, $x \in \Delta_{2^{n-r}}$.
Regarding "$=$" in (8) as the generalization of biconditional operator $\leftrightarrow$. converts (8) into $M_{=}(M_I \otimes I_{2^r})xy = \delta_2^1$.
(8) can be solved from Boolean equation (7), if and only if there exists $M_I \in \mathcal{L}^{2^r \times 2^{n-r}}$, satisfying

$$L_I = M_{=} M_I. \tag{9}$$

D. Cheng, X. Xu. Automatica, 2013.
Y. Qiao, H. Qi and D. Cheng. IEEE T. Neur. Net. Lear., 2018.
S. Wang, J. Feng,Y. Yu, X. Wang. Sci. China: Inform. Sci., 2020.

Example 4-State Feedback Stabilization

- State-feedback stabilizer

$$u(t) = Kx(t), \ K \in \mathcal{L}^{2^m \times 2^n} \tag{10}$$

Substituting (10) into $\mathcal{B}(L, H)$, converts it into a Boolean network with structure matrix $LK\Phi_{2^n}$.

- Stabilize $\mathcal{B}(L, H)$ to set Ξ

Gain matrix K can be calculated by

$$LK\Phi_{2^n} = F, \tag{11}$$

where F is dependent on L and Ξ.

- Input Observability \ Nonsingualrity $\Longrightarrow$ Inverse Systems
- Block Decoupling $\Longrightarrow$ Structure Matrices of Subsystems
- Time-invariant Output Feedback Stabilization $\Longrightarrow$ Gain Matrix

- Fuzzy Relational Equation
- Potential Equation
- Matrix Equation on $\mathcal{R}$

D. Cheng, J. Feng, H. Lv, IEEE T. Fuzzy Syst., 2012.
H. Li, Y. Wang. Automatica, 2013.
D. Cheng. Automatica, 2014.
X. Liu, J. Zhu. Automatica, 2016.
Y. Yu, J. Feng, J. Pan, D. Cheng. IEEE T. Auto. Contr., 2019.
Y. Yu, B. Wang, J. Feng. Neurocomputing, 2019.
H. Fan, J. Feng, M. Meng, B. Wang. Fuzzy Set. Syst., 2020.
Z. Ji, J. Li, X. Zhou, F, Duan, T, Li. Linear. Multilinear A., 2021.
J. Wang. Electron. Res. Arch., 2024.

Canonical Solutions of Logical Matrix Equations

Matrix equation (11) can be generalized to a class of logical matrix equation,

$$AXB = C, \tag{12}$$

where $A \in \mathcal{L}^{m_1 \times n_1}$, $B \in \mathcal{L}^{p_2 \times q_1}$, $C \in \mathcal{L}^{m_2 \times q_2}$ are known, and $X \in \mathcal{L}^{n_2 \times p_1}$ are unknown variables.

Definition 2

X is a canonical solution of logical matrix equation (12), if $X \in \mathcal{L}^{n_2 \times p_1}$ satisfies equation (12).

Under admissible dimensions, equation (12) can be expressed as

$$(A \otimes I_{\frac{m_2}{m_1}})(X \otimes I_{\tau})(B \otimes I_{\frac{q_2}{q_1}}) = C. \tag{13}$$

Canonical Solutions of Logical Matrix Equations

Proposition 1

Canonical solutions set of logical matrix equation (12) is $\{X \in \mathcal{L}^{n_2 \times p_1} | X \preceq S_\tau(X)\}$, where $M_{\bar{X}} = A^T \ltimes \mathrm{Sgn}(C \ltimes B^T), \tau = \frac{q_2 \cdot p_2}{q_1 \cdot p_1}$,

$$\mathrm{Col}_i(\tilde{M}_{\bar{X}}) = \begin{cases} \mathrm{Col}_i(M_{\bar{X}}), & \text{if } \mathrm{Col}_i(C \ltimes B^T) \neq \mathbf{0}, \\ \mathbf{1}_{n_2 \tau}, & \text{otherwise}, \end{cases} \tag{14}$$

$$[S_\tau(X)]_{i,j} = \prod_{k=1}^{\tau} [(\delta_{n_2}^i)^T \tilde{M}_{\bar{X}} \delta_{p_1}^j]_{k,k}. \tag{15}$$

Two classes of logical matrix equations:

$$AX = B, \ A \in \mathcal{L}^{m\times n}, \ B \in \mathcal{L}^{m\times p} \tag{16}$$

$$XB = C, \ B \in \mathcal{L}^{p\times q}, \ C \in \mathcal{L}^{n\times q} \tag{17}$$

Proposition 2

- The logical matrices $X \in \mathcal{L}^{n\times p}$, satisfying (16), constitute the set $\{X \in \mathcal{L}^{n\times p} | X \preceq A^\top B\}$.
- The logical matrices $X \in \mathcal{L}^{n\times p}$, satisfying (17), constitute the set $\{X \in \mathcal{L}^{n\times p} | X \succeq \mathrm{Sgn}(CB^T)\}$.

Y. Yu, J. Feng, J. Pan, D. Cheng. IEEE T. Auto. Contr., 2019.

Example 5-Transition Matrix of Singular Boolean Network

The algebraic expression of a singular Boolean network with n nodes is presented as follows:

$$Ex(t+1) = Fx(t), \tag{15}$$

where $E, F \in \mathcal{L}^{2^n \times 2^n}$ and $x(t+1), x(t) \in \Delta_{2^n}$.

Proposition 3

Assuming that the adjacency matrix of the state transition graph for (15) is A_G, we have $A_G = E^T F$.

J.Feng, J.Yao, P.Cui. Sci. China Inform. Sci., 2013.
M.Meng, J.Feng. IET Control Theory A., 2014.
Y. Liu, B. Li , H.Chen, J. Cao. Automatica, 2017.
Y.Yu, J.Feng, M.Meng, B.Wang. IET Control Theory A., 2017.

Example 6-Invariant Subspaces of BCN

$\mathcal{Z}$ is called an $Lu-$invariant subspace of $\mathcal{B}(L, H)$, if there exist $M_i \in \mathcal{L}^{2^r \times 2^r}$, $i = 1, 2, \ldots, 2^m$, such that

$$GF_i = M_i G, \tag{18}$$

where structure matrix of subspace $\mathcal{Z} = \mathcal{F}_l\{z_1, z_2, \ldots, z_r\}$ is $G \in \mathcal{L}^{2^r \times 2^n}$.

Proposition 4

Subspace $\mathcal{Z}$ is an $Lu-$invariant subspace of $\mathcal{B}(L, H)$, if and only if $\mathbf{1}^T \mathrm{Sgn}(GF_i G^T) \preceq \mathbf{1}^T$, $i = 1, 2, \ldots, 2^m$.

D. Cheng, L. Zhang and D. Bi. IEEE T. Auto. Contr., 2022.

Section III: Application: Convergent Matrix Sequence Method

Subsection III.i: Observability

Definition 3

BCN (3) is said to be observable, if for any $x_0 \neq x_0' \in \Delta_{2^n}$, there exists input $u(t)$, such that $[y(x_0, u(0)), y(x_0, u(1)), \dots] \neq [y(x_0', u(0)), y(x_0', u(1)), \dots]$.

Definition 3'

BCN (3) is not observable, if there exist states $x_0 \neq x_0' \in \Delta_{2^n}$, such that $[y(x_0, u(0)), y(x_0, u(1)), \dots] = [y(x_0', u(0)), y(x_0', u(1)), \dots]$ for any input $u(t)$.

For any $u(t) := \delta_{2^m}^{k_t}$,

$$\begin{cases} H\delta_{2^n}^{\alpha} & = & H\delta_{2^n}^{\beta}, \\ \vdots \\ H\prod_{s=\tau-1}^{1} F_{k_s}\delta_{2^n}^{\alpha} & = & H\prod_{s=\tau-1}^{1} F_{k_s}\delta_{2^n}^{\beta}, \\ \vdots \end{cases} \tag{19}$$

The distinguishability matrix $\mathcal{O} = \lim_{i\to\infty} \bar{\mathcal{M}}_i$, where $\bar{\mathcal{M}}_0 = H^\top H$,

$$\bar{\mathcal{M}}_{i+1} = \bigwedge_{k=1}^{2^m} F_k^\top \bar{\mathcal{M}}_i F_k \wedge \bar{\mathcal{M}}_i.$$

Theorem 1

BCN (3) is observable if and only if $\mathcal{O} = I_{2^n}$.

Y. Yu, M. Meng, J. Feng, G. Chen. IEEE T. Auto. Contr., 2022.

Probabilistic Boolean network:

$$\begin{cases} x(t+1) & = F_{\sigma(t)}x(t), \\ y(t) & = Hx(t), \end{cases} \tag{20}$$

where $x(t) \in \Delta_{2^n}$, $y(t) \in \Delta_{2^p}$. Here we suppose that $\sigma(\cdot)$ satisfies $\mathbf{P}(\sigma(t) = i) = p_i$, $t > 0$, where $\sum_{i=1}^{s} p_i = 1$ and $p_i > 0$.

Definition 4

Probabilistic Boolean network (20) is said to be observable, if there exists an integer T, such that for every admissible output sequence $\mathbf{y}(x_0, \sigma, T)$, it is possible to uniquely identify the corresponding initial state of (20).

R. Zhou, Y. Guo, and W. Gui. Automatica, 106:230–241,2019.

E. Fornasini, M. E. Valcher, IEEE Contr. Syst. Lett., 4(2): 319–324, 2020.

Theorem 2

Probabilistic Boolean network (20) is observable if and only if $\lim_{i\to\infty} \mathcal{M}_i = I_{2^n}$, where $\mathcal{M}_0 = H^\top H$, $\mathcal{M}_{i+1} = \mathrm{Sgn}(L_{\mathcal{B}}^\top \mathcal{M}_i L_{\mathcal{B}}) \wedge \mathcal{M}_i$, $L_{\mathcal{B}} = \vee_{i=1}^{s} F_i$.

Y. Yu, M. Meng, J. Feng, G. Chen. IEEE T. Auto. Contr., 2022.

C. Wang, J. Feng, Y. Yu. Syst. Contr. Lett., 174:105485, 2023.

Section III.ii: Observability Decomposition

Definition 5

The observability decomposition of BCN $\mathcal{B}(L, H)$ is implementable, if there is a coordinate transformation $z(t) = Tx(t)$, under which (3) can be equivalently transformed into

$$\begin{cases} z^{[1]}(t+1) & = & G_1 u(t) z^{[1]}(t), \\ z^{[2]}(t+1) & = & G_2 u(t) z(t), \\ y(t) & = & M z^{[1]}(t), \end{cases} \tag{21}$$

where $z(t) = z^{[1]}(t) z^{[2]}(t)$, $z^{[i]}(t) \in \Delta_{2^{n_i}}$, $n_1 + n_2 = n$, and subsystem $\mathcal{B}(G_1, M)$ is observable.

Theorem 3

The observability decomposition of (3) is implementable, if and only if there exists an integer $0 < s < n$, such that $\mathcal{O}\mathbf{1}_{2^n} = 2^s \mathbf{1}_{2^n}$.

Y. Li, J. Zhu. IEEE T. Auto. Contr., 2022.

Y. Yu, C. Wang, J. Feng, G. Chen. IEEE T. Auto. Contr., 2024.

Subsection III.iii: Detectability

Definition 6

BCN (3) is detectable, if there exist an integer $s > 0$ and an input sequence $u(0), u(1), \ldots, u(s-1)$ such that for any different initial states $x_0, x_0' \in \Delta_{2^n}$, $x(x_0, u(\cdot), s) \neq x(x_0', u(\cdot), s)$ and $Hx_0 = Hx_0'$ imply $(Hx_0, \ldots, Hx(x_0, u(\cdot), s)) \neq (Hx_0', \ldots, Hx(x_0', u(\cdot), s))$.

Theorem 4

BCN (3) is detectable, if and only if there exists an integer $s > 0$ such that $\mathbf{0}_{2^n \times 2^n} \in S(\mathcal{M}^s)$, where $S(\mathcal{M}^0) = \{H^\top H \wedge \Upsilon\}$ and $\Upsilon = \mathbf{1}_{2^n \times 2^n} - I_{2^n}$,

$$S(\mathcal{M}^{i+1}) = \{F_k^\top \mathcal{M} F_k \wedge H^\top H \wedge \Upsilon |\ \mathcal{M} \in S(\mathcal{M}^i), 1 \leq k \leq 2^m\}.$$

K. Zhang, L. Zhang, L. Xie, Discrete-time and Discrete-space Dynamical Systems, Springer, 2020.

B. Wang, J. Feng, H. Li, Y. Yu. Nonlinear Anal-Hybr., 2020.

C. Wang, J. Feng, Y. Yu. J. Franklin I., 2024.

Subsection III.iv: Fault Detectability

Definition 7

Set that $x_0, \tilde{x}_0 \in \Delta_{2^n}$ are initial states of $\mathcal{B}(L, H)$ and $\mathcal{B}(\tilde{L}, \tilde{H})$, respectively. BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is said to be active fault-detectable, if there exists an integer $T \in \mathbb{N}$, such that

$$\begin{aligned} & (y(x_0, u(0)), y(x_0, u(1)), \ldots, y(x_0, u(T))) \\ \neq \ & (\tilde{y}(\tilde{x}_0, u(0)), \tilde{y}(\tilde{x}_0, u(1)), \ldots, \tilde{y}(\tilde{x}_0, u(T))). \end{aligned} \tag{22}$$

holds for any input sequence $\{u(t)\}|_{t=0}^{+\infty}$ and any $x_0, \tilde{x}_0 \in \Delta_{2^n}$.

Theorem 5

BCN $\mathcal{B}(\tilde{L}, \tilde{H})$ is active fault-detectable if and only if

$$\lim_{i \to \infty} \hat{\mathcal{M}}_i = \mathbf{0}_{2^n \times 2^n}, \tag{23}$$

where $\hat{\mathcal{M}}_{i+1} = \bigvee_{k=1}^{2^m} F_k^\top \hat{\mathcal{M}}_i \tilde{F}_k \wedge \hat{\mathcal{M}}_i$, $\hat{\mathcal{M}}_0 = H^\top \tilde{H}$.

R. Zhao, C. Wang, Y. Yu, J. Feng. Sci. China: Inform. Sci., 2023.

Section IV: Conclusion

- Give canonical solution sets of logical matrix equations. Applications: Coloring problem, Implicit function theorem, Stabilization, Invariant subspaces, Decoupling, etc.

Y. Yu, J. Feng, J. Pan, D. Cheng. IEEE T. Auto. Contr., 2019.
Y. Yu, M. Meng, J. Feng. Automatica, 2021.
Y. Yu, M. Meng, J. Feng, G. Chen. IEEE T. Auto. Contr., 2022.
C. Wang, J. Feng, Y. Yu. Syst. Contr. Lett., 2023.
R. Zhao, C. Wang, Y. Yu, J. Feng. Sci. China: Inform. Sci., 2023.
C. Wang, J. Feng, Y. Yu. J. Franklin I., 2024.
Y. Yu, C. Wang, J. Feng, G. Chen. IEEE T. Auto. Contr., 2024.

- Give canonical solution sets of logical matrix equations. Applications: Coloring problem, Implicit function theorem, Stabilization, Invariant subspaces, Decoupling, etc.
- The method based on convergent matrix sequences is effective and unified to solve a series of problems. Observability (Decomposition), Realization, Identification, (Fault) Detectability, etc.

Y. Yu, J. Feng, J. Pan, D. Cheng. IEEE T. Auto. Contr., 2019.
Y. Yu, M. Meng, J. Feng. Automatica, 2021.
Y. Yu, M. Meng, J. Feng, G. Chen. IEEE T. Auto. Contr., 2022.
C. Wang, J. Feng, Y. Yu. Syst. Contr. Lett., 2023.
R. Zhao, C. Wang, Y. Yu, J. Feng. Sci. China: Inform. Sci., 2023.
C. Wang, J. Feng, Y. Yu. J. Franklin I., 2024.
Y. Yu, C. Wang, J. Feng, G. Chen. IEEE T. Auto. Contr., 2024.

Thanks for your attention!

Q & A